A STUDY OF THE DYNAMICS OF MACHINE FOUNDATIONS

by

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DEPARTMENT OF CIVIL ENGINEERING

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A STUDY
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OF MACHINE FOUNDATIONS

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ABSTRACT

The design of foundations for machines with rotational speeds below about 1,800 R.P.M. usually involves careful consideration of natural frequencies of foundation and soil systems. Often the amplitude of vibration of the foundation is of importance, however, a theoretical determination of this amplitude can be expected to give only an order of magnitude. Consequently, this thesis is mainly concerned with the study of factors which influence the natural frequencies of machine foundations.

It is found that consideration of the natural frequency for one mode of vibration may be insufficient and for most foundations more than one degree of freedom should be investigated. The determination of natural frequencies in some machine foundation problems is not possible by analytical methods. Certain factors which influence natural frequencies are presented and if these are considered, it should be possible to design so that resonance is avoided.

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I N D E X

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1. INTRODUCTION

It has been found (1)* that the natural frequencies of most foundation soil systems are less than 30 c.p.s., consequently there is the possibility of resonance occurring if the rotational speed of a machine mounted on a soil supported foundation is less than 1,800 R.P.M. Most stationary reciprocating machines have rotational speeds less than 1,800 R.P.M., and consideration of the dynamic characteristics of the foundations of such machines is necessary in order that resonance be avoided. Dangerous fatigue stresses and other objectionable factors, such as excessive vibration transmitted to buildings and other adjacent structures are usually present when the foundation is at or near resonance.

Analytical methods, along with the qualitative consideration of certain factors not amenable to analysis can provide a basis for at least an approximate determination of the natural frequencies of the various modes of vibration. However, determination of amplitudes of vibration of foundations is highly uncertain by analytical methods.

A rigid foundation is a system with six possible degrees of freedom, that is, rotations about a vertical axis and two mutually perpendicular horizontal axes and translations in the directions of these three axes. If a foundation cannot be considered rigid, then more degrees of freedom are introduced. Each degree of freedom has a

*References are given in Bibliography.

natural frequency associated with it, and it is desirable that the rotational speed or twice the rotational speed of the machine should not be within approximately 20% of any one of these natural frequencies. The term "natural frequency" is used with some misgiving, as a soil-foundation system displays non-linear characteristics, and any natural frequency of the system is not a constant but a function of the disturbing force (10). This indicates that linear theories are not strictly applicable to analysis of the dynamic behaviour of soil systems; however, linear theories can be used if the magnitude of the disturbing force is small compared with the weight of the machine and foundation.

Satisfactory foundations for low speed machines usually have natural frequencies which are higher than the rotational speed of the machine (2). Foundations for high speed machines, such as gas and steam turbines with rotational speeds greater than 3,000 R.P.M. have natural frequencies below the rotational speed of the machine. The two types of machine, therefore, give rise to rather different foundation design problems.

2. A THEORETICAL ANALYSIS OF SOIL VIBRATIONS

(a) Simple Spring Supported Mass Analogy

Several procedures have been suggested to determine foundation natural frequencies (3). Perhaps the most obvious procedure is to consider the foundation as a single degree of freedom spring supported mass, with the soil being assumed a massless spring. The natural frequency is then given by:-

$$f = \frac{1}{2\pi}\sqrt{\frac{Kg}{W_V}} \qquad (2.1)$$

where W_V = weight of foundation and machine

K = spring constant (restoring force per unit displacement)

g = dimensional gravitational constant.

This procedure is a gross oversimplification and Eq. 2.1 gives unsatisfactory results. One refinement consists of making an allowance for the soil which vibrates in phase with the foundation by considering this soil dynamically equivalent to a mass of weight W_S, vibrating with the same amplitude as the foundation (4). The natural frequency is then given by:-

$$f = \frac{1}{2\pi}\sqrt{\frac{k'Ag}{W_S+W_V}} \qquad (2.2)$$

where k' = coefficient of dynamic subgrade reaction (restoring force per unit area for unit displacement)

A = area of foundation bearing on soil

$k'A$ = stiffness (restoring force per unit displacement)

W_V = weight of foundation and machine.

The determination of f is dependent on the values of k' and W_S.

The value of W_S can only be obtained from the results of experiments. It has been found (1) that W_S is usually the dominant weight in Eq. 2.2.

Lorenz (10) assumed k' was constant for a given soil, however, Tschebotarioff and Ward (3) have shown that this is incorrect. The following procedure is suggested to determine an approximate value for k' for vertical displacement.

Consider the foundation to be rigid. The vertical deflection of a rigid foundation, resting on an elastic medium, is very nearly equal to the average deflection of a uniformly loaded surface of the same dimensions with the same total load (12). This deflection is given by (5):-

$$W = \frac{mP(1-\nu^2)}{E\sqrt{A}} \qquad (2.3)$$

where m = constant dependent on ratio of length to width

P = Load

A = contact area = ab

a = width

b = length

ν = Poisson's ratio of supporting medium

E = modulus of elasticity of supporting medium.

The coefficient m is found from Fig. 1 which is plotted from data given in reference 12.

The coefficient of dynamic subgrade reaction can then be found from

$$k' = \frac{E'}{m(1-\nu^2)\sqrt{A}} \qquad (2.4)$$

E has been replaced by E' the dynamic modulus of elasticity. For soils the dynamic modulus has not the same value as the static modulus (12).

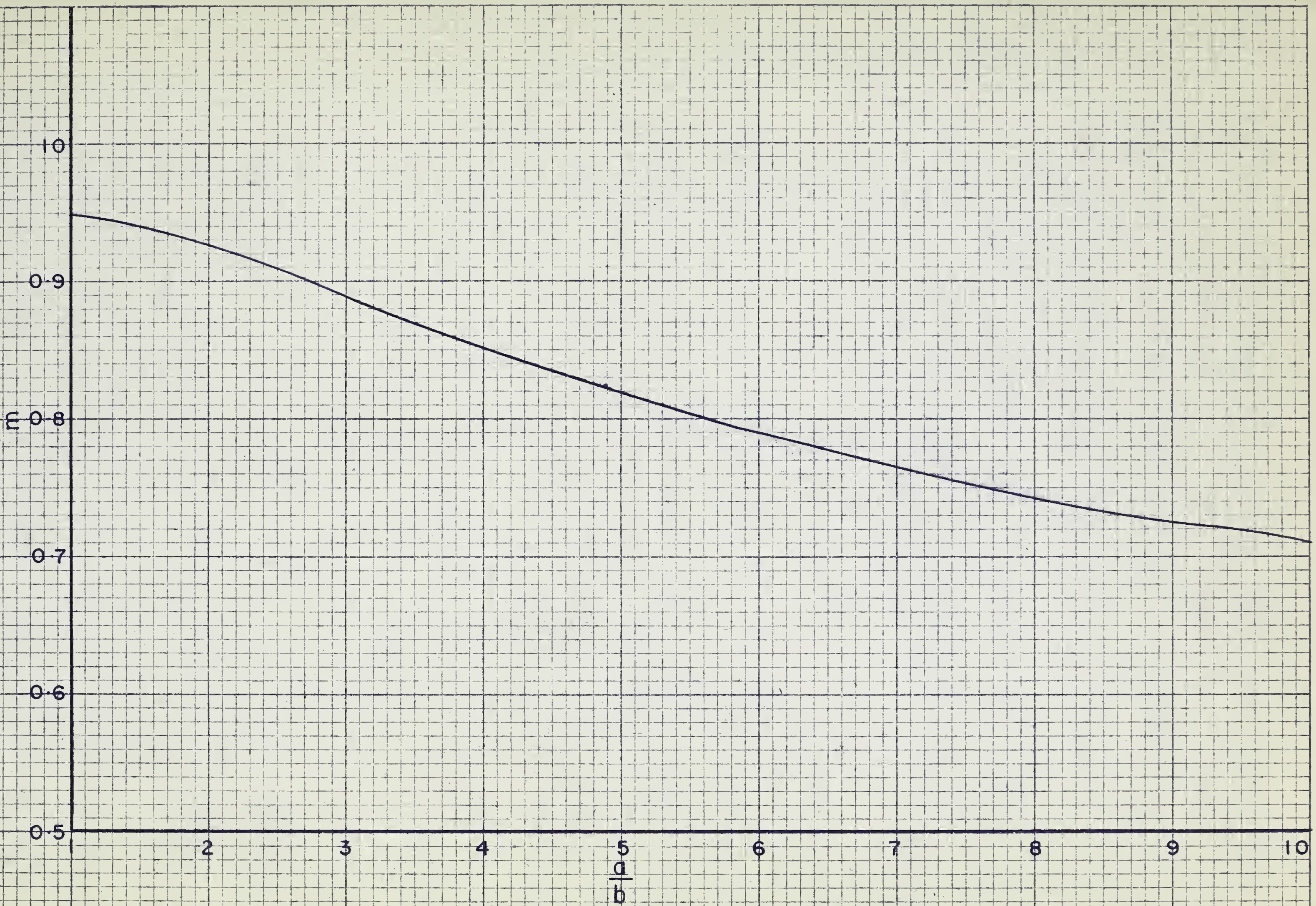
It is evident from Eq. 2.4 that k' is not a constant for a given soil, but is inversely proportional to the square root of the contact area.

Apart from the indeterminate nature of W_s and k', Eq. 2.2 has the disadvantage that it gives the natural frequency for only the vertical degree of freedom, and often other degrees of freedom may have to be considered.

(b) Bearing Pressure

The average bearing pressure, $\frac{W_v}{A}$, must be given careful consideration by the designer of foundations for low speed machines.

It has been found (2) that the lower the bearing pressure, the higher is the natural frequency of the foundation on a given soil. This indicates that if the area of a foundation is increased and thus the bearing pressure reduced the stiffness (k'A) will be increased more than the total vibrating weight (W_s+W_v). The static modulus of subgrade reaction is increased when



COEFFICIENT FOR EQUATION 2·4 FIG.1

the bearing pressure is reduced. It is therefore reasonable to assume that the dynamic modulus will likewise be increased when the bearing pressure is reduced.

(c) Analogous Damped, Spring Supported Mass with Disturbing Force.

When the periodic force is vertical sinusoidal, and acts such that the foundation undergoes only vertical displacement, then the amplitude frequency characteristics of the foundation are similar to those of a damped, single degree of freedom, spring supported mass (2). Soil vibrators used for experimental work are an example of this (12). Most machine foundations have more complicated amplitude-frequency relations due to the several degrees of freedom and also the higher harmonics of the periodic disturbing forces. However, some investigators have studied foundations with amplitude-frequency characteristics similar to those of a single degree of freedom, damped, spring supported mass. This suggests that the concept of a spring supported mass with viscous damping is useful for qualitative consideration of the dynamic behaviour of machine foundations. For example, the effect of adding mass to a foundation may be studied qualitatively.

Consider the analogous single degree of freedom, damped, spring supported mass $\frac{W}{g}$ acted on by a periodic force, $Q\omega^2 \text{Sin } \omega t$. The unbalanced forces produced by machines are usually proportional to the square of the rotational speed. The amplitude of the disturbing force

is therefore taken as $Q\omega^2$, where Q is dimensionally equivalent to mass times length and ω is the angular frequency of the machine. The amplitude of vibration is given by:-

$$y_o = \frac{\frac{Qp^2}{K} \cdot \frac{\omega^2}{p^2}}{\sqrt{\left(1 - \frac{\omega^2}{p^2}\right)^2 + \frac{4n^2\omega^2}{p^4}}} \qquad (2.5)$$

where K = stiffness of the spring

$p^2 = Kg/W$

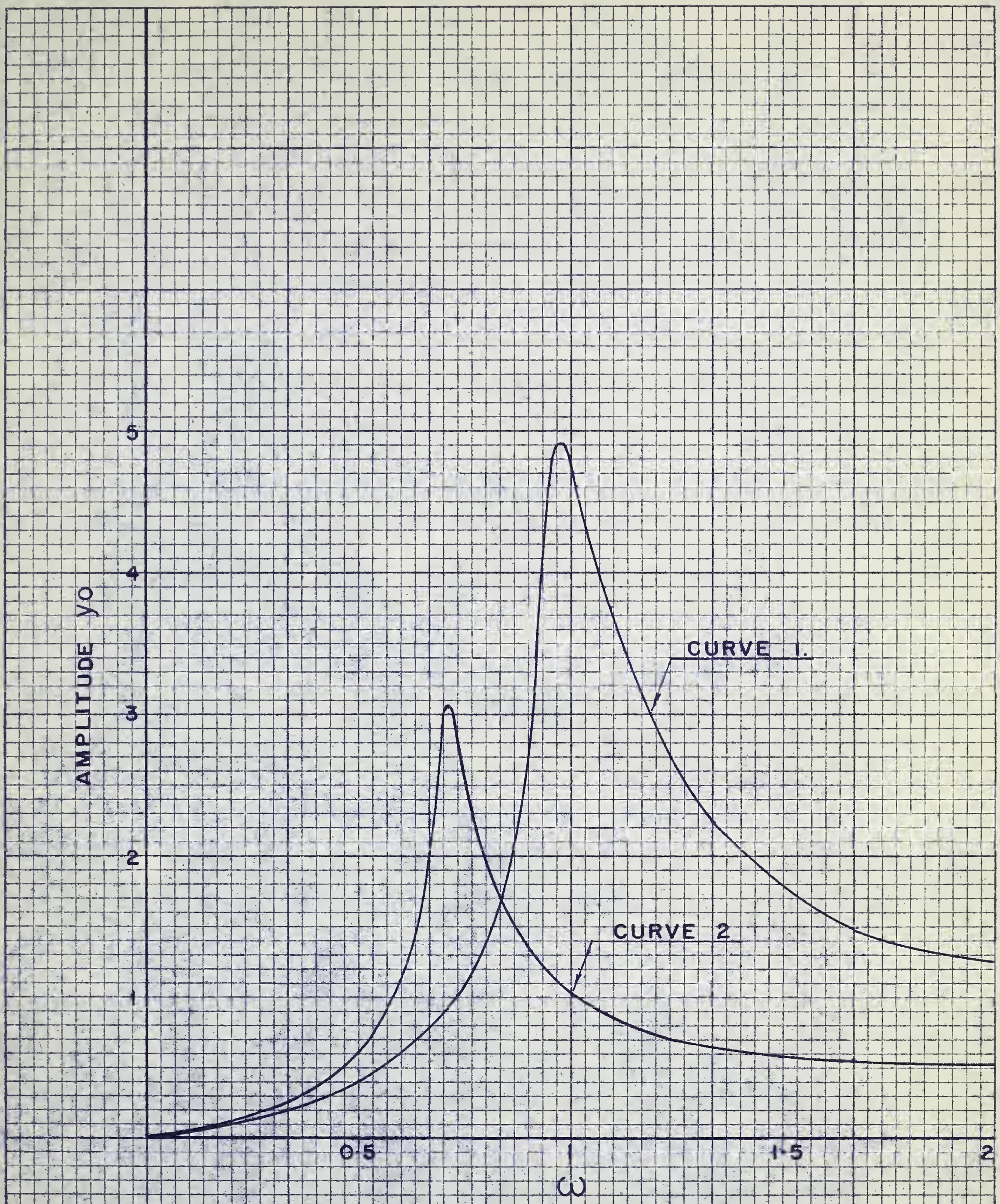
$2n = Cg/W$

C = viscous damping constant

g = dimensional gravitational constant.

Curve 1, Fig. 2, shows the variation of amplitude with frequency for $W/g = 1$, $p = 1$, $Q = 1$ and $C = 0.2\sqrt{KW/g}$. This damping constant is one tenth of the critical damping constant. Resonance occurs when $\omega = p\sqrt{1 - \frac{n^2}{p^2}}$ which is approximately equal to p. Curve 2, Fig. 2, shows the effect of doubling the mass and keeping the remaining terms constant. This indicates that if the machine speed is in a certain range an increase in the weight of the foundation may increase the amplitude of vibration.

The effect of damping is negligible when the speed of the machine is less than half the resonance speed for the foundation, and Newcomb (2) suggests that Eq. 2.5 may then be used to find the vibration amplitude. However, the weight W in Eq. 2.5 should be taken as $W_s + W_v$.



AMPLITUDE CURVES FOR SPRING SUPPORTED MASS

FIG.2

(d) Approximate Method for Determining Foundation Natural Frequencies.

The approximate procedures outlined in this section are based on Rayleigh's Principle and may be used to find the natural frequencies for all of the six degrees of freedom of rigid foundations. Analytical determination of natural frequencies for rotational vibration about various axes is usually not practicable because of the unavailability of the values of mass moments of inertia of machines. For this reason the application of the method to vertical and horizontal vibration, only, is given here.

For determination of the natural frequency of vertical vibration the assumptions and procedure are as follows:

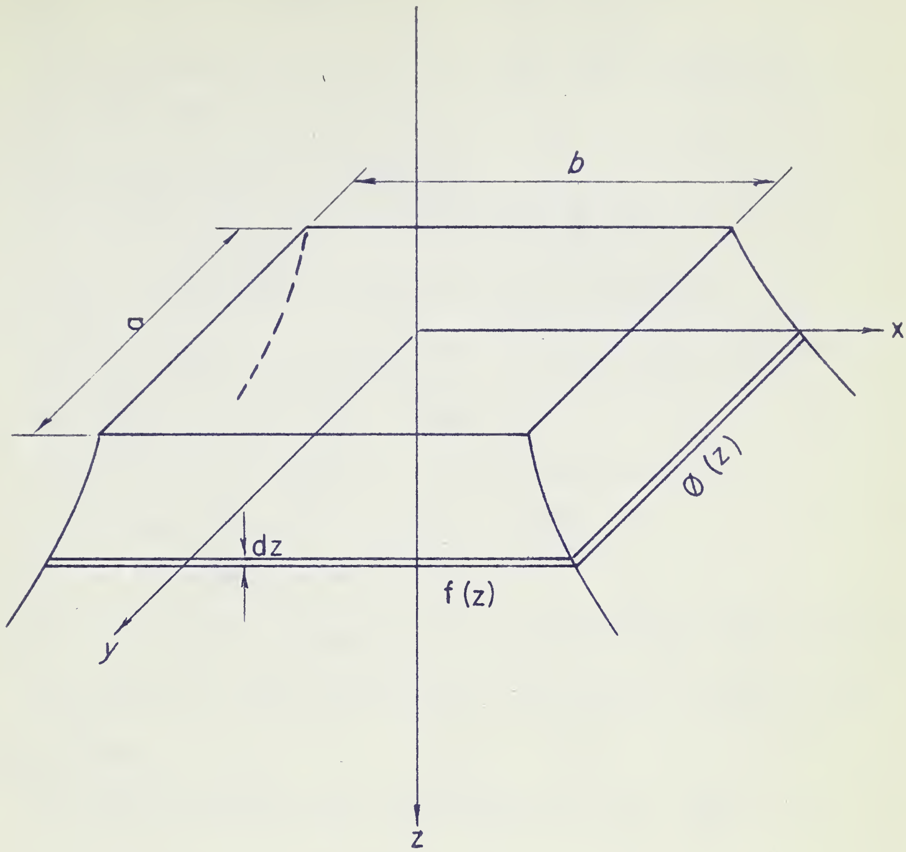
Assumptions:-

(1) The system may be considered conservative in order to determine natural frequencies.

(2) Dynamic pressure is transmitted through soil contained in a solid formed by the base of the foundation and the surfaces $y = f(z)$, $y = -f(z)$, $x = \phi(z)$, and $x = -\phi(z)$ as shown in Fig. 3.

(3) The dynamic stress at any depth z is uniformly distributed over a section of the solid parallel to the (xy) plane, that is parallel to the base of the foundation.

This last assumption is inaccurate but is useful for the development of relations which are at best dependent on empirical data.



MODEL FOR APPROXIMATE METHOD

FIG.3

The notation is as follows:-

M = mass of foundation and machine

ρ = density of soil (mass per unit volume)

a = width of foundation

b = length of foundation

A = bearing area of foundation = ab

W_{of} = amplitude of vibration of foundation

f_z^{xy} = natural frequency for vertical vibration

ω = angular natural frequency

E' = dynamic modulus of elasticity of soil

G = dynamic shear modulus of soil

ν = Poisson's ratio of soil

σ_z = vertical normal stress (dynamic) in z direction.

Consider an elemental layer of soil at depth z as shown in Fig. 3.

Let the amplitude of vibration of this layer = W_o.

W_o decreases as z increases.

Let W_o be given by:-

$$W_o = W_{of} e^{-\beta z}$$

β is dimensionally equivalent to L^{-1}.

Assume foundation vibrates sinusoidally:-

$$W_f = W_{of} \operatorname{Sin} \omega t$$

The maximum kinetic energy of the foundation and machine is given by:-

$$T' = \tfrac{1}{2} M\omega^2 W_{of}^{\,2}$$

It is now necessary to find T" the maximum kinetic energy of the vibrating soil.

Let amplitude of dynamic strain at $z = 0$ be ϵ_{zo}.

$$\epsilon_{zo} = \left|\frac{\partial W_o}{\partial z}\right|_{z=o}$$

$$= \beta W_{of}$$

Let s be area of layer of elemental thickness at depth z.

Amplitude of dynamic strain at z is given by:-

$$\epsilon_z = \left|\frac{\partial W_o}{\partial z}\right|$$

$$= \beta W_{of} e^{-\beta z}$$

Equating the force acting on contact area ab, to the force acting on elemental layer gives:-

$$E'\beta W_{of} e^{-\beta z} s = \beta W_{of} abE'$$

$$\therefore s = abe^{\beta z}$$

Assume the elemental layer vibrates as follows:-

$$W = W_o \text{ Sin } \omega t$$

The maximum kinetic energy of the elemental layer is given by:-

$$dT'' = \tfrac{1}{2}\rho abe^{\beta z}\omega^2 W_o{}^2 dz$$

$$= \tfrac{1}{2}\rho abe^{-\beta z} W_{of}{}^2 \omega^2 dz$$

$\therefore$ The maximum kinetic energy of the soil is

$$T'' = \int_0^{\infty} dT''$$

$$= \tfrac{1}{2}\rho abW_{of}^{2}\omega^{2}\int_{0}^{\infty} e^{-\beta z}dz$$

$$= \frac{\rho abW_{of}^{2}\omega^{2}}{2\beta}$$

The maximum kinetic energy of the system is then given by:-

$$T = T' + T''$$

$$= \frac{\rho abW_{of}^{2}\omega^{2}}{2\beta} + \frac{M\omega^{2}W_{of}^{2}}{2}$$

$$= \frac{W_{of}^{2}\omega^{2}}{2}\left[\frac{\rho ab}{\beta} + M\right]$$

Consider the strain energy of the system.

The amplitude of $\sigma_z = E'\left|\frac{\partial W_o}{\partial z}\right|$

The maximum strain energy of the elemental layer is then given by:-

$$dV = \frac{|\sigma_z|^2}{2E'} \text{ X Vol. of layer}$$

$$= \tfrac{1}{2}E'\left(\frac{\partial W_o}{\partial z}\right)^{2} abe^{\beta z}dz$$

$$= \tfrac{1}{2}E'\beta^{2}W_{of}^{2}e^{-\beta z}abdz$$

Maximum strain energy is then given by:-

$$V = \int_{0}^{\infty} dV$$

$$= \tfrac{1}{2}E'\beta^{2}W_{of}^{2}ab\int_{0}^{\infty} e^{-\beta z}dz$$

$$= \tfrac{1}{2}E'\beta W_{of}^{2}\, ab$$

For a conservative system, according to Rayleigh's

Principle, the maximum strain energy is equal to the maximum kinetic energy.

i.e. $V = T$

$$\therefore \tfrac{1}{2}E'\beta W_{of}^{2}ab = \frac{W_{of}^{2}\omega^{2}}{2}\left[\frac{\rho ab}{\beta} + M\right]$$

$$\omega = \sqrt{\frac{E'\beta ab}{\frac{\rho ab}{\beta} + M}}$$

$$\therefore f_z^{xy} = \frac{1}{2\pi}\sqrt{\frac{E'\beta ab}{\frac{\rho ab}{\beta} + M}}$$

$$G = \frac{E'}{2(1+\nu)}$$

$$\therefore f_z^{xy} = \frac{1}{2\pi}\sqrt{\frac{2G(1+\nu)\beta ab}{\frac{\rho ab}{\beta} + M}} \qquad (2.6)$$

The determination of f_z^{xy} is dependent on the evaluation of β. For highly compacted sand the constant may be taken as:

$$\beta = \frac{1.7}{m\sqrt{A}\,(1-\nu^2)} \ \text{ft.}^{-1} \qquad (2.7)$$

Where m is given in Fig. 1 and A is in sq. ft.

It has not been possible to obtain a dependable expression for β for foundations on clay. However, if the foundation of Section b, Part 5 of this thesis is considered, and the dynamic value of G for clay is taken as 1500 p.s.i. and $\nu = 0.5$, then β is found to be given by:

$$\beta = \frac{2.2}{m\sqrt{A}\,(1-\nu^2)}$$

Eq. 2.7 is found by equating the deflection of the loaded surface of the model shown in Fig. 3 to a constant times the deflection given by Eq. 2.3 for the same load. The constant is evaluated by applying the results of vibrator tests given in references 1 and 8. (See Appendix)

Eq. 2.6 may be applied to a soil vibrator with a circular base if ab is replaced by the area of the base. In Eq. 2.7 m is then taken as one.

Two examples of the application of Eq. 2.6 are now given. Quinlan (7) describes a rectangular vibrator 5 ft. by 3 ft., weighing 12,500 lbs. and resting on sand with G = 2,000 p.s.i. (found from soil vibrator tests), $\nu = 1/3$ and $\rho = 100$ lbs. per cu. ft.

From Eq. 2.7:-

$$\beta = \frac{1.7}{0.93 \times 3.87 \times 0.89}$$

$$= 0.53 \text{ ft.}^{-1}$$

From Eq. 2.6:-

$$f_z^{xy} = \frac{1}{2\pi}\sqrt{\frac{2\text{x}2{,}000\text{x}144\text{x}(1+0.33)\text{x}0.53\text{x}5\text{x}3\text{x}32.2}{\frac{100\text{x}5\text{x}3}{0.53} + 12{,}500}}$$

$$= 16 \text{ c.p.s.}$$

The actual natural frequency is given in reference 7 as 14 c.p.s., however, the amplitude of the disturbing force was 9,000 lb. and for a disturbing force small

compared with the weight of the vibrator the natural frequency would be higher. The value found from Eq. 2.6 may therefore be considered satisfactory for practical considerations.

A further example is taken from Table I of reference 11. The area of the foundation is 10 sq. ft. and the weight of the foundation and machine is 2.8 tons. Properties of the sand are assumed to be the same as in the previous example.

From Eq. 2.7:

$$\beta = \frac{1.7}{1 \text{x} 3.16 \text{x} 0.89}$$

$$= 0.602 \text{ ft.}^{-1}$$

From Eq. 2.6:

$$f = \frac{1}{2\pi}\sqrt{\frac{2\text{x}2{,}000\text{x}144\text{x}(1+0.33)\text{x}0.602\text{x}10\text{x}32.2}{\frac{100\text{x}10}{0.602} + 5{,}600}}$$

$$= 23.2 \text{ c.p.s.}$$

$$= 1390 \text{ c.p.m.}$$

Table I of reference 11 gives the natural frequency as 1445-1600 c.p.m. The result obtained from Eq. 2.6 is therefore accurate enough for design purposes.

In order to consider the natural frequency for horizontal vibration in the x direction of Fig. 3, the same assumptions as before are made along with the assumption that the dynamic shearing stress is uniformly distributed over a section of the solid parallel to the

xy plane. The procedure is then as follows, with the notation the same as before except as noted.

Let the horizontal amplitude of foundation vibration $= U_{of}$

Let the horizontal amplitude of elemental soil layer at any depth z $= U_o$

U_o decreases as z increases.

Assume $U_o = U_{of}e^{-\beta z}$

Amplitude of shear strain at z is

$$\gamma_{xz} = \frac{\partial U_o}{\partial z} + \frac{\partial W}{\partial x}$$

Where W is displacement in z direction when displacement in x direction is U_o.

Assume W = 0

$$\gamma_{xz} = \frac{\partial U_o}{\partial z}$$

At z = 0

$$\gamma_{xz} = \left| \frac{\partial U_o}{\partial z} \right|_{z = 0}$$

$$= \beta U_{of}$$

$\therefore$ Amplitude of restoring force $= G\beta U_{of}ab$

Let area of elemental layer = S

$\therefore$ Amplitude of restoring force $= G \left| \frac{\partial U_o}{\partial z} \right| S$

$$= G\beta U_{of}e^{-\beta z}S$$

But $G\beta U_{of}ab = G\beta U_{of}e^{-\beta z}S$

$\therefore \quad S = abe^{\beta z}$

Assume the vibration of the foundation and soil is simple harmonic thus:-

$$U_f = U_{of} \operatorname{Sin} \omega t$$

$$U = U_o \operatorname{Sin} \omega t$$

The maximum kinetic energy of the foundation and machine is given by:-

$$T' = \frac{M}{2}\left(\frac{\partial U_f}{\partial t}\right)^2$$

$$= \tfrac{1}{2} M U_{of}^{2}\omega^2$$

The maximum kinetic energy of elemental layer is given by:-

$$dT'' = \tfrac{1}{2} \rho S dz \left(\frac{\partial U}{\partial t}\right)^2$$

$$= \tfrac{1}{2}\rho a b e^{\beta z}\omega^2 U_o^2 dz$$

$$= \tfrac{1}{2}\rho a b \omega^2 U_{of}^2 e^{-\beta z} dz$$

The maximum kinetic energy of the soil is then given by:-

$$T'' = \int_0^\infty dT''$$

$$= \tfrac{1}{2}\rho a b \omega^2 U_{of}^2 \int_0^\infty e^{-\beta z} dz$$

$$= \frac{\rho a b \omega^2 U_{of}^2}{2}$$

T = the total kinetic energy.

$$T = T' + T''$$

$$= \tfrac{1}{2}MU_{of}^{\,2}\omega^2 + \frac{\rho ab\omega^2 U_{of}^{\,2}}{2\beta}$$

The maximum strain energy of the soil is given by:-

$$V = \int_0^\infty \frac{\{G\gamma_{xz}S\}^2 dz}{2GS}$$

$$= \int_0^\infty \frac{G\gamma_{xz}^{\,2}S}{2}\,dz$$

$$= \int_0^\infty \frac{G}{2}\left(\frac{\partial U_o}{\partial z}\right)^2 S\,dz$$

$$= \frac{G}{2}\int_0^\infty U_{of}^{\,2}e^{-\beta z}\beta^2 ab\,dz$$

$$= \frac{G}{2}U_{of}^{\,2}\beta ab$$

For a conservative system, according to Rayleigh's Principle, the maximum strain energy is equal to the maximum kinetic energy.

i.e. $V = T$

$$\tfrac{1}{2}MU_{of}^{\,2}\omega^2 + \frac{\rho ab\omega^2 U_{of}^{\,2}}{2\beta} = \frac{G}{2}U_{of}^{\,2}\beta ab$$

$$\therefore \quad \omega = \sqrt{\frac{Gab\beta}{\frac{\rho ab}{\beta}+M}}$$

$$\therefore \quad f_x^{xy} = \tfrac{1}{2}\pi\sqrt{\frac{Gab\beta}{\frac{\rho ab}{\beta}+M}} \qquad (2.8)$$

Where f_x^{xy} = natural frequency for vibrations in the x direction.

It is evident that $f_x^{xy} = f_y^{xy}$.

Where f_y^{xy} = natural frequency for vibrations in

the y direction.

Newcomb (2) states that the natural frequency for horizontal vibrations is usually very nearly equal to the vertical natural frequency. This was found to be true for the unmodified foundation considered in Section (b) of Part 5 of this thesis. It is, therefore, evident that if Eqs. 2.6 and 2.8 are compared, β is probably greater for horizontal vibration than for vertical vibration.

(e) Modes of Vibration

Pauw (9) has outlined a method to determine the natural frequencies of the various degrees of freedom of a rigid foundation. In this method the system is assumed conservative and the Lagrangian Equations are used. In practice this may be a needless refinement since it is possible that certain of the six modes of vibration of a rigid foundation may be capable of existence alone without the presence of any other modes.

A mode of vibration which is capable of independent existence is termed an uncoupled mode. Similarly a mode which cannot exist without the presence of one or more different modes is termed a coupled mode. For example, the vertical vibration of a foundation is an uncoupled mode if a steady vertical force with its line of action passing through the centre of gravity of the machine and foundation produces vertical displacement with no

rotation or horizontal displacement. If a couple is applied with respect to a specified axis through the centre of gravity of the machine and foundation and rotation takes place about the axis with no displacement of the centre of gravity, the rotational mode is uncoupled. If a mode is uncoupled, the natural frequency for this mode may be calculated as for a single degree of freedom system.

For most machine-foundation systems it is probable that the condition for vertical vibration being uncoupled is met at least approximately. Usually it is not practicable to calculate the natural frequencies for rotational modes because of the unavailability of values of the moments of inertia of machines about the required axes.

Newcomb (2) states that for shallow foundations the natural frequency for rotation about a longitudinal axis is approximately equal to the vertical natural frequency.

(f) Non Linear Effects

Soil vibrator test results show that soil-vibrator systems have non-linear vibrational characteristics, the natural frequency being a function of the disturbing force (10). The greater the disturbing force the lower is the natural frequency. This indicates that k' the coefficient of dynamic subgrade reaction decreases with

increased displacement, as does the static coefficient which is sometimes known as the foundation modulus.

Soil vibrators used to demonstrate non-linear behaviour have disturbing forces of the same order of magnitude as the weight of the vibrator. However, for most stationary machines the disturbing forces due to unbalance of rotating and reciprocating parts are small compared with the weight of the machine and foundation.

When the disturbing force is small compared with the weight of the machine and foundation it is reasonable to assume that k' is very nearly independent of the dynamic displacement, and that vibrational characteristics are very close to being linear.

(g) Wave Propagation

The most fundamental approach to foundation vibration problems is to consider the foundation to be resting on part of the surface of a semi-infinite isotropic elastic medium. This has been done by Quinlan (7) and Sung (8). This approach is of practical value since it shows that a soil foundation system is not a conservative system even if the soil is assumed to be an ideal elastic solid with no damping. For example, if part of the surface of a semi-infinite elastic solid with no damping is given a small displacement and released the amplitude of vibration of a point on the surface will decrease with time. This is due to waves which are propagated in

every direction from the region of the disturbance. Propagation of these waves requires energy and hence the system is not conservative. The dissipation of energy produces an effect similar to that of viscous damping, consequently it is permissible to assume the system is conservative when approximate methods such as those of Section (d) are used to determine natural frequencies.

The equations of motion of an elastic solid form the basis for the work of Quinlan (7) and Sung (8). These equations in Cartesian coordinates are as follows:-

$$(\lambda + G)\frac{\partial e}{\partial x} + G\nabla^2 U - \rho\frac{\partial^2 U}{\partial t^2} = 0 \qquad (2.9)$$

$$(\lambda + G)\frac{\partial e}{\partial y} + G\nabla^2 V - \rho\frac{\partial^2 V}{\partial t^2} = 0 \qquad (2.10)$$

$$(\lambda + G)\frac{\partial e}{\partial z} + G\nabla^2 W - \rho\frac{\partial^2 W}{\partial t^2} = 0 \qquad (2.11)$$

Where U = displacement in x direction

V = displacement in y direction

W = displacement in z direction

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

e = dilatation or volume expansion

λ = Lame's constant

$$= \frac{\nu E}{(1+\nu)(1-2\nu)}$$

These equations can be used to study wave propagation in a medium of linear stress-strain characteristics.

If Eq. 2.9 is differentiated with respect to x, and Eq. 2.10 with respect to y, and Eq. 2.11 with respect to z and added, the result is:-

$$\rho \frac{\partial^2 e}{\partial t^2} = (\lambda + 2G)\nabla^2 e$$

Put $\dfrac{\lambda + 2G}{\rho} = a_1{}^2$

$$\therefore \quad \nabla^2 e = \frac{1}{a_1{}^2} \frac{\partial^2 e}{\partial t^2} \qquad (2.12)$$

This is the well known wave equation and shows the dilatation e is propagated through the material with velocity $a_1 = \sqrt{\dfrac{\lambda + 2G}{\rho}}$.

If e is eliminated from Eqs. 219, 2.10 and 2.11 and the substitution made:-

$$2\bar{\omega}_x = \frac{\partial W}{\partial y} - \frac{\partial V}{\partial z}$$

then the following result is obtained:-

$$\nabla^2 \bar{\omega}_x = \frac{1}{a_2{}^2} \frac{\partial^2 \bar{\omega}_x}{\partial t^2} \qquad (2.13)$$

where $a_2 = \sqrt{\dfrac{G}{\rho}}$

This is again a wave equation and shows that the rotation $\bar{\omega}_x$ is propagated with velocity $a_2 = \sqrt{\dfrac{G}{\rho}}$. Similarly rotations $\bar{\omega}_z$ and $\bar{\omega}_y$ are propagated with velocity a_2.

So far it has been shown that two types of waves radiate from a disturbance in an elastic medium. The

waves given by Eq. 2.12 are usually called "waves of dilatation", although the term "extensional waves" is sometimes used and the waves given by Eq. 2.13 are usually called "waves of distortion". Kolsky (13) points out that the waves given by Eq. 2.12 should be termed irrotational and those given by Eq. 2.13 should be termed equivoluminal.

In an unbounded solid the above two types of waves, only, are propagated. When there is a bounding surface, however, elastic surface waves occur. The most important type of surface waves is known as Rayleigh waves. The effect of Rayleigh waves decreases rapidly with depth and the particles move in elliptical orbits in planes parallel to the direction of wave propagation (13). Rayleigh waves have a velocity slightly less than that of waves of distortion. For example, when $\nu = 0.25$ the velocity of Rayleigh waves is $0.9194\sqrt{\frac{G}{\rho}}$, and when $\nu = 0.5$ it is $0.9553\sqrt{\frac{G}{\rho}}$, whereas the velocity of distortion waves is $\sqrt{\frac{G}{\rho}}$.

Seismographic records of the waves obtained some distance from earthquakes indicate that the amplitude of Rayleigh waves is large compared with the amplitudes of dilatation and distortion waves (13). It is probable that a similar condition exists near a vibrating machine foundation and that vibration is transmitted to surrounding structures mainly by Rayleigh waves.

Quinlan's analysis (7) leads to the following expression for the natural frequency f of a mass resting

on a circular part of the surface of a semi-infinite isotropic elastic solid:-

$$f = \frac{a_o}{2\pi a} \sqrt{\frac{G}{\rho}} \qquad (2.14)$$

where a = radius of the circular part of the surface.

$$a_o = f(b)$$

$$b = \frac{M}{2\rho a^3}$$

$$M = \text{mass}$$

Quinlan's paper gives a plot of a_o against b. For solids such as steel, the dynamic value of G to be used in expressions such as Eq. 2.14 is very close to the value found from static tests. For soils this is not true (12). The natural frequency of a soil vibrator on a given soil can be found experimentally and used in Eq. 2.14 to find the dynamic value of G for the soil.

Soil is not an elastic isotropic solid, however, Eq. 2.14 suggests that the natural frequency of a soil-foundation system is a function of the mass of the foundation and machine, the density of the soil, the area of the foundation and the velocity of propagation of distortion waves. A study of the results of soil vibrator tests shows that there is a relationship between the natural frequency and the velocity of wave propagation in the soil, the higher the wave velocity, the higher is

the natural frequency. A study of Table I of reference 1 will show this. One example is the effect on the natural frequency of a soil-foundation system, of raising or lowering the water table. When the water table rises the natural frequency of the soil-foundation system increases (3) as does the velocity of wave propagation (14). Laugharne Thornton (14) comments on the sensitivity to vibration of "water logged" soil. It follows that the velocity of propagation of one of the wave types is a property which could be useful when the design of a machine foundation is considered. The velocity of distortion waves or of Rayleigh waves is suggested as a design consideration rather than the velocity of dilatation waves because of the dependency of the latter on Poisson's Ratio (4).

The dynamic value of G found from soil vibrator tests by applying Eq. 2.14 is less than that found from the wave velocities of the soil. This is a consequence of the non-linear deformation characteristics of the soil. The modulus found from wave propagation is a tangent modulus because of the low stresses produced by wave propagation. The dynamic stresses produced by soil vibrators are superimposed upon relatively large static stresses and therefore the modulus found from soil vibrator tests is not a tangent modulus.

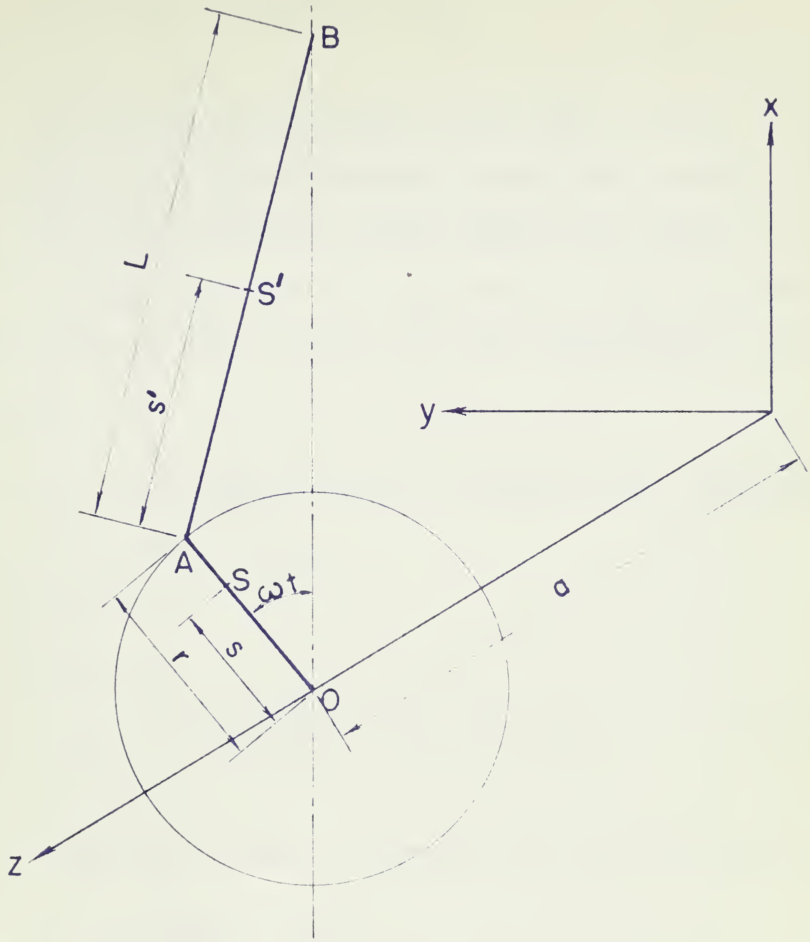
3. DISTURBING FORCES AND MOMENTS PRODUCED BY RECIPROCATING MACHINES

Vibration of machine foundations is caused by disturbing forces and moments due to the moving masses of the machines. As the origin of a vibration problem is the agency causing the vibration, it is desirable that the nature of any disturbing forces and moments produced by a machine should be known at least qualitatively if a foundation is to be designed for the machine.

Disturbing forces and moments produced by a reciprocating machine are inertia effects. Internal forces and moments caused by gas pressure on the pistons, friction in the mechanisms, etc., can produce no disturbing effects, since there is always an equal and opposite force or moment (17). It is assumed that cyclic variations in crankshaft speed are small.

In order to consider the disturbing forces and moments due to a single cylinder mechanism, three axes must be defined. Let the z axis be the axis of the crankshaft, the x axis parallel to the line of the stroke and the y axis perpendicular to the xz plane as shown in Fig. 4. The choice of the letters to designate these axes is arbitrary and the letters may be interchanged if convenient.

For a single cylinder mechanism the components in the x, y and z directions of the inertia forces and moments are X, Y, Z, and M_x, M_y, M_z, respectively, and are as follows:



SCHEMATIC
DIAGRAM of RECIPROCATING MECHANISM

FIG. 4

X is the force in the line of stroke;

Y is the transverse force;

Z is zero;

M_x is the yawing moment about the x axis;

M_y is the pitching moment about the y axis;

M_z is the rolling moment about the z axis.

It can be shown (17) that for a single cylinder mechanism as shown in Fig. (4) and operating at constant R.P.M.

$$X = \omega^2 \left[(Q+Q') \text{ Cos } \omega t + Q(A_2 \text{Cos } 2\omega t - A_4 \text{Cos } 4\omega t + \ldots .) \right]$$

$$Y = \omega^2 Q' \text{ Sin } \omega t$$

$$M_x = Ya$$

$$M_y = Xa$$

$$M_z = \omega^2 R(C_1 \text{ Sin } \omega t - C_3 \text{ Sin } 3\omega t \text{ ------------})$$

$$Z = 0$$

Where ωt = crank angle measured from top dead centre;

$$Q = \frac{r}{g}\left[\frac{s'}{l}G' + G''\right] ,$$

$$Q' = \frac{1}{g}\left[sG + r(1 - \frac{s'}{l})\ G'\right] ,$$

$$R = \frac{\lambda}{g}\left[s'\ (l-s') - K'^2\right] G'$$

$$\lambda = \frac{r}{l}$$

G = Weight of crank, crank pin and crank web, etc.

G' = Weight of the connecting rod.

G'' = Weight of reciprocating parts.

s = Distance OS in Fig. (7) where S is the centre of gravity of crank.

s' = Distance AS' where S' is the centre of gravity of connecting rod.

K' = Radius of gyration of the connecting rod about an axis through S' parallel to the crank axis.

A_2 .... A_{2n} are power series in λ.

C_1, C_3...... C_{2n+1} are power series in λ.

Most modern engines have the crank webs designed such that Q' is zero, then Y becomes zero. For most purposes it is sufficiently accurate to take the above expressions for X and M_z when Q' is zero as,

$$X = \omega^2 Q \left[\cos \omega t + \lambda \operatorname{Cos} 2\omega t\right]$$

$$M_z = \omega^2 R \left[\operatorname{Sin} \omega t - \frac{3}{8}\lambda^2 \operatorname{Sin} 3\omega t\right]$$

These expressions indicate that the disturbing force X in the line of stroke can be considered to have a first and second harmonic, usually known as the primary and secondary components. Similarly, the rolling moment M_z has a first and third harmonic. The rolling moment M_z is usually not considered in elementary treatises on engine balancing but is of considerable importance. It is possible to design the connecting rod such that R and hence M_z is zero, but this is very rarely done.

For multi-cylinder engines the forces and moments for the various cylinders are combined. As with the single cylinder mechanism it is usually necessary to consider

only first, second and third harmonics as the amplitudes of higher harmonics are negligible.

The expressions for the disturbing forces and moments for multi-cylinder engines are given in reference 17.

Moments M_x and M_y for a multi-cylinder engine are computed with respect to an origin on the crank axis and midway between the centre lines of the outermost cylinders.

4. MEASUREMENT OF VIBRATION

Vibration can be measured in terms of displacement, velocity and acceleration. The simplest vibration is simple harmonic motion, the displacement y being given by:-

$$y = A \sin \omega t \qquad (4.1)$$

where A is the amplitude and ω is the angular frequency. For a simple harmonic vibration the velocity v is given by:-

$$v = \frac{dy}{dt} = A\omega \cos \omega t \qquad (4.2)$$

and the acceleration a is given by:-

$$a = \frac{d^2y}{dt^2} = -A\omega^2 \sin \omega t \qquad (4.3)$$

The waveforms of most vibrations encountered in practice are not simple harmonic, as they consist of a fundamental harmonic and higher harmonics. Displacement y is then given by a series thus:-

$$y = C_0 + C_1 \sin(\omega t+\alpha_1) + C_2 \sin(\omega t+\alpha_2) + \ldots\ldots$$

$$\ldots\ldots + C_n \sin(\omega t+\alpha_n) + \ldots\ldots (4.4)$$

Terms $C_1 \sin(\omega t+\alpha_1)$, $C_2 \sin(\omega t+\alpha_2)$, etc., are called the first or fundamental harmonic, second harmonics, etc., respectively, and α_1, α_2, etc., are the phase angles. If the disturbing force producing the vibration is F Sin ωt then the maximum value of the component $C_1 \sin(\omega t+\alpha_1)$ occurs at an interval of time $\frac{\alpha_1}{\omega}$ before the maximum value

of F Sin ωt occurs. Term C_0, which does not influence the shape of the wave, is dependent on the position of the t axis and may be eliminated with proper choice of this axis. Eq. 4.4 may be rewritten as a Fourier series with sine and cosine terms.

For simple harmonic vibrations if the displacement amplitude, the velocity amplitude and the acceleration amplitude or any two of these amplitudes are known, then the angular frequency can be found. For example, if a and y are known where a is the acceleration and y is the displacement, then from Eqs. 4.1 and 4.3:-

$$\frac{|a|}{|y|} = \frac{A\omega^2}{A} = \omega^2$$

This procedure is not possible for other wave forms, as the higher harmonics are progressively more important in velocity and acceleration.

A General Radio Company Type 761 vibration meter may be used to measure vibration of foundations. This meter employs a Rochelle Salt crystal as a pick-up. The instrument can be used to find the root-mean-square amplitude of sinusoidal vibration in terms of displacement, velocity and acceleration. For wave forms other than sinusoidal, the readings can be taken as root-mean-square amplitudes with a high degree of accuracy. For a sine wave the amplitude is $\sqrt{2}$ times the root-mean-square amplitude and this relationship is close for most wave forms except extreme cases such as square waves. The

exact root-mean-square amplitude $\bar{y}$ of periodic wave forms other than sinusoidal is given by:-

$$\bar{y} = \sqrt{\frac{C_1^{\,2} + C_2^{\,2} + \ldots\ldots C_n^{\,2} + \ldots\ldots}{2}} \qquad (4.5)$$

where C_1 ..... C_n are as in Eq. 4.4.

Further information on wave forms is given in reference 16.

The vibration meter may be used with the General Radio Company Type 762 vibration analyser, as shown in Fig. 6. This analyser gives the amplitudes of the various harmonics, that is, the constants C_1, C_2 .....C_n, of Eq. 4.4. It is not possible to draw the wave form from the information obtained from the analyser since the phase angles α_1......α_n..... are not known. A cathode ray oscilloscope may be used with the meter and the actual wave form can then be observed.

Further information on the vibration meter and analyser is given in reference 15.

5. EXPERIMENTAL STUDIES

(a) Behavior of Foundation of Three Cylinder G.M.C. Diesel Engine.

Tests were made to determine the amplitude - R.P.M. relationships for an engine foundation. The engine is situated in the Mechanical Engineering Laboratory of the University of Alberta and is shown in Fig. 5. The foundation is of concrete and is detailed in Fig. 7. Relevant data for the engine, the dynamometer and the foundation are as follows:-

Soil .... Clay

Weight of foundation .... 25,000 lbs.

Weight of engine .... 1,150 lbs.

Weight of dynamometer .... 900 lbs.

Bearing area of foundation .... 69 sq. ft.

Bearing pressure .... 390 lb. per sq. ft.

Number of cylinders of engine 3

Cycle .... 2 Stroke

Bore and stroke .... $4\frac{1}{2}$ x 5 in.

Governed R.P.M. 1600

Maximum B.H.P. 65

Firing Interval .... 120°

$\lambda = \frac{r}{L}$ = Crank length/connecting rod length = 0.259

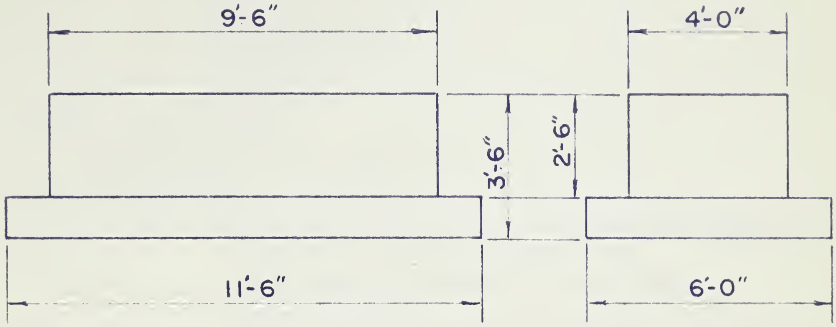
Crank length r .... 2.5 in.

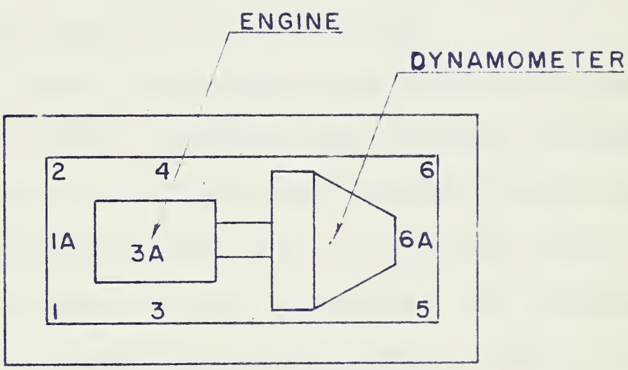
Disturbing forces and couples due to engine:-

$$\left.\begin{array}{ll}\text{Vertical disturbing force} & - 0 \\ \text{Horizontal disturbing force} & - 0\end{array}\right\}\begin{array}{l}\text{Both primary} \\ \text{and secondary}\end{array}$$

G.M.C. DIESEL ENGINE AND DYNAMOMETER Fig. 5

VIBRATION METER AND ANALYSER Fig. 6





PLAN SHOWING MEASUREMENT POSITIONS

DIESEL ENGINE FOUNDATION

FIG. 7

Primary pitching couple -- $a\sqrt{3}\, m\omega^2 \sin \omega t$

Secondary pitching couple -- $a\sqrt{3}\, m\omega^2 \lambda \sin 2\omega t$

where $m = \frac{1}{g}\left[\frac{s'}{L}G' + G''\right]$

$= 0.243$ Slugs

i.e. $Q = m\, r$

(Notation as in Part 3)

a = distance between the centre lines of adjacent cylinders = 5.8 in.

Rolling couple $= 3\omega^2 R \left[\frac{3}{8}\lambda^2 \sin 3\omega t + \text{higher harmonics which are negligible.}\right]$

(R is defined in Part 3)

It was not possible to evaluate R.

The above data indicate that any vibration caused by the engine is due to primary and secondary pitching couples acting in a vertical plane passing through the axis of the crankshaft, and also to a rolling couple acting about the axis of the crankshaft. The frequency of this couple is three times the angular speed of the engine. Examination of the dynamometer indicates that the rotor is out of balance. This means there is vertical and horizontal disturbing forces and couples acting on the foundation in addition to those due to the engine. Components of the forces and couples due to the dynamometer are sinusoidal and should not give rise to higher harmonics.

The vibration measurements were made with a General Radio Company Type 761 vibration meter and the Type 762 analyser was used to obtain the displacement amplitudes of the higher harmonics at certain points on the foundation. At engine speeds greater than 600 R.P.M. a cathode ray oscilloscope was used with the vibration meter to examine the wave form. The "pick up" was placed on the foundation and held with two bricks as shown in Fig. 6.

Two significant results of the tests should be noted before the R.P.M. - amplitude curves are considered. These results are as follows:-

(1) The amplitude of displacement at any given R.P.M. for a point on the foundation is independent of the load on the engine. Also when the engine is motored by using the dynamometer as an electric motor, the amplitude - R.P.M. characteristics are the same as when the engine is under load. This indicates that the foundation vibration is caused entirely by inertia forces and moments due to the unbalance of the rotating and reciprocating parts of the engine and dynamometer.

(2) At engine speeds at which the peaks of the R.P.M. - amplitude curves occur the wave form is almost a pure sine curve. This was shown with a cathode ray oscilloscope.

Curves of displacement amplitude against R.P.M. are shown in Figs. 8 to 20. An arrow on the sketch shown with a graph refers to horizontal vibration in the direc-

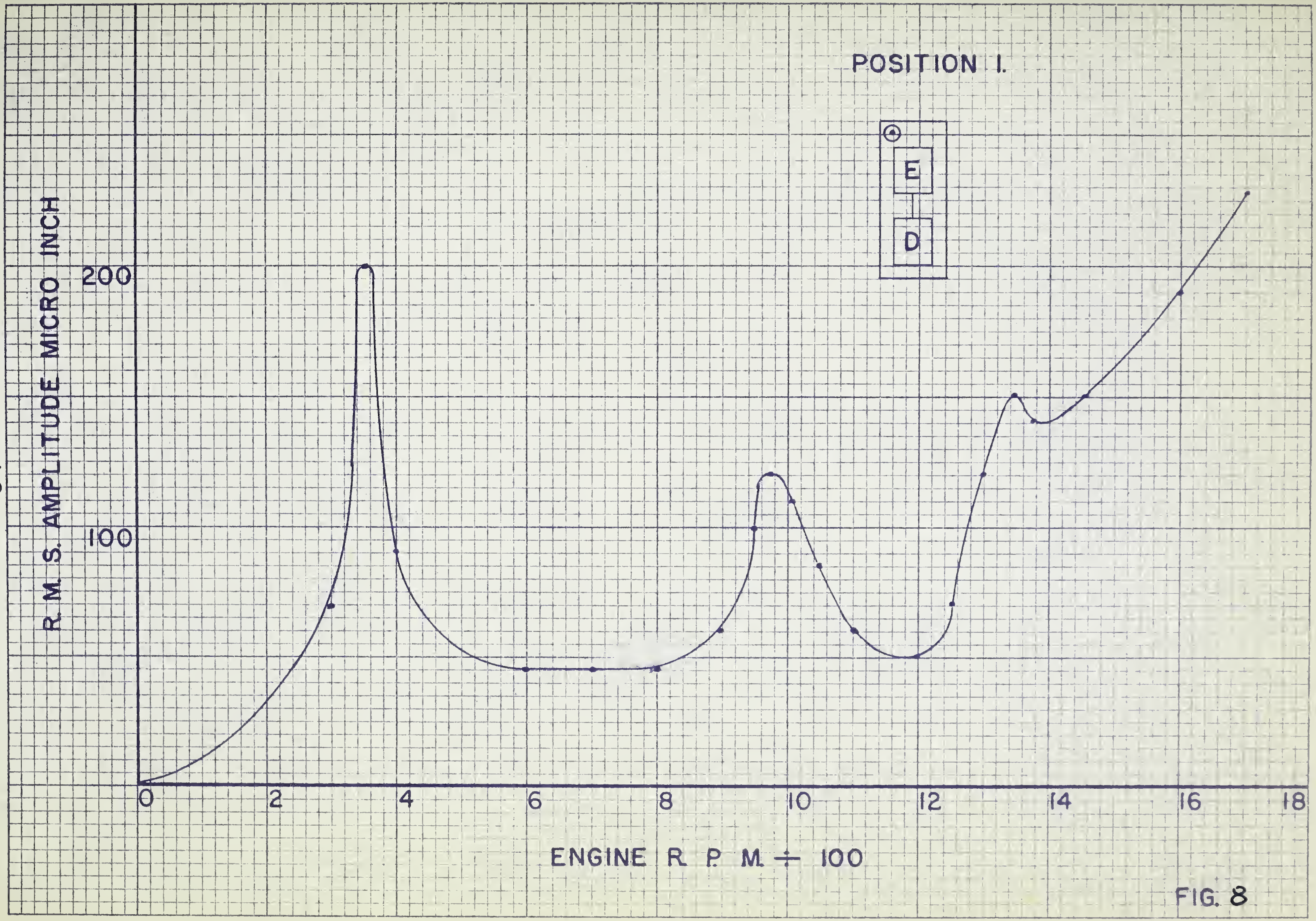

POSITION I.
E
D
R. M. S. AMPLITUDE MICRO INCH
200
100
0
2
4
6
8
10
12
14
16
18
ENGINE R. P. M. ÷ 100

FIG. 8

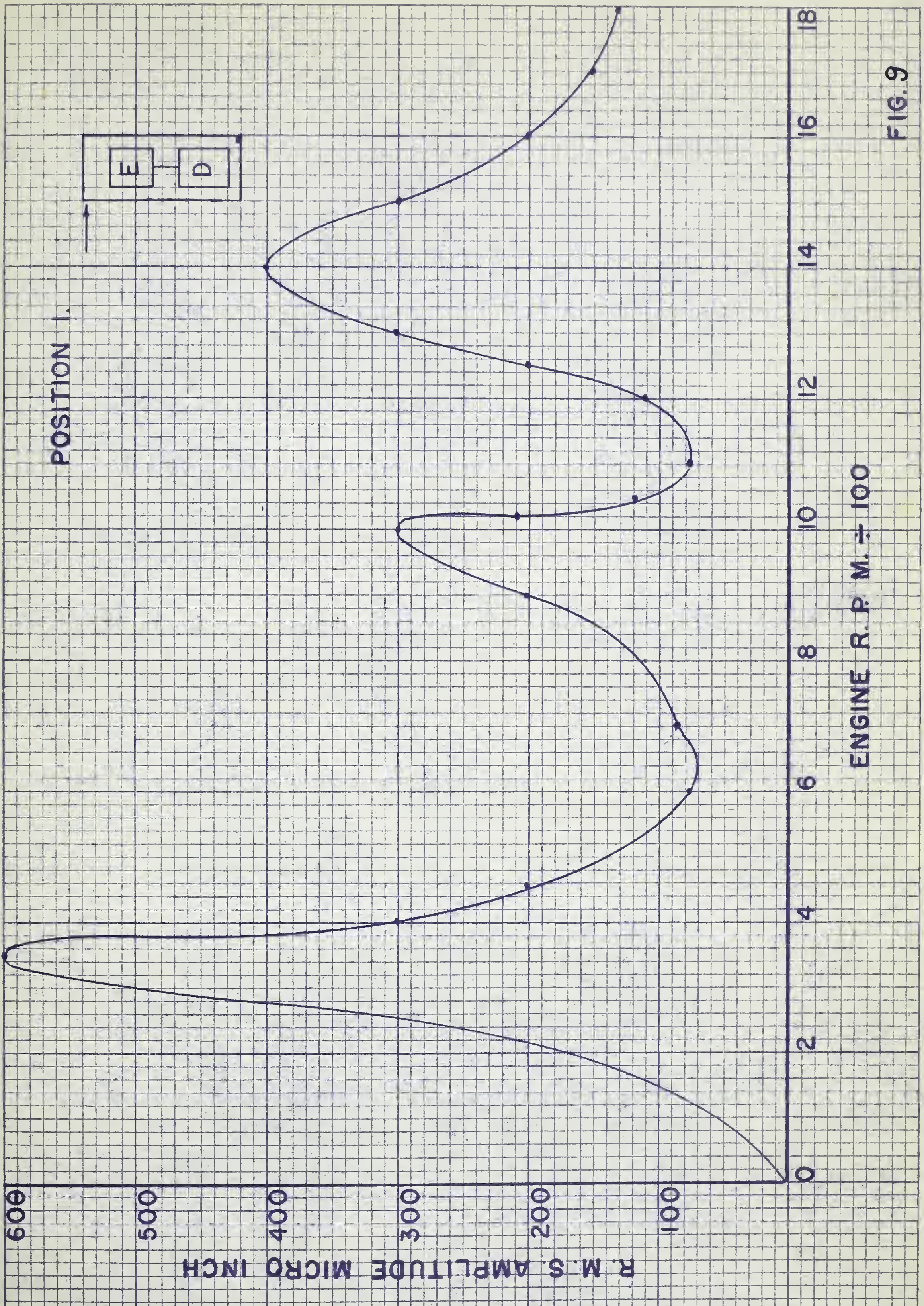

POSITION 1.
E
D
R.M.S. AMPLITUDE MICRO INCH
600
500
400
300
200
100
0
2
4
6
8
10
12
14
16
18
ENGINE R.P.M. ÷ 100

FIG. 9

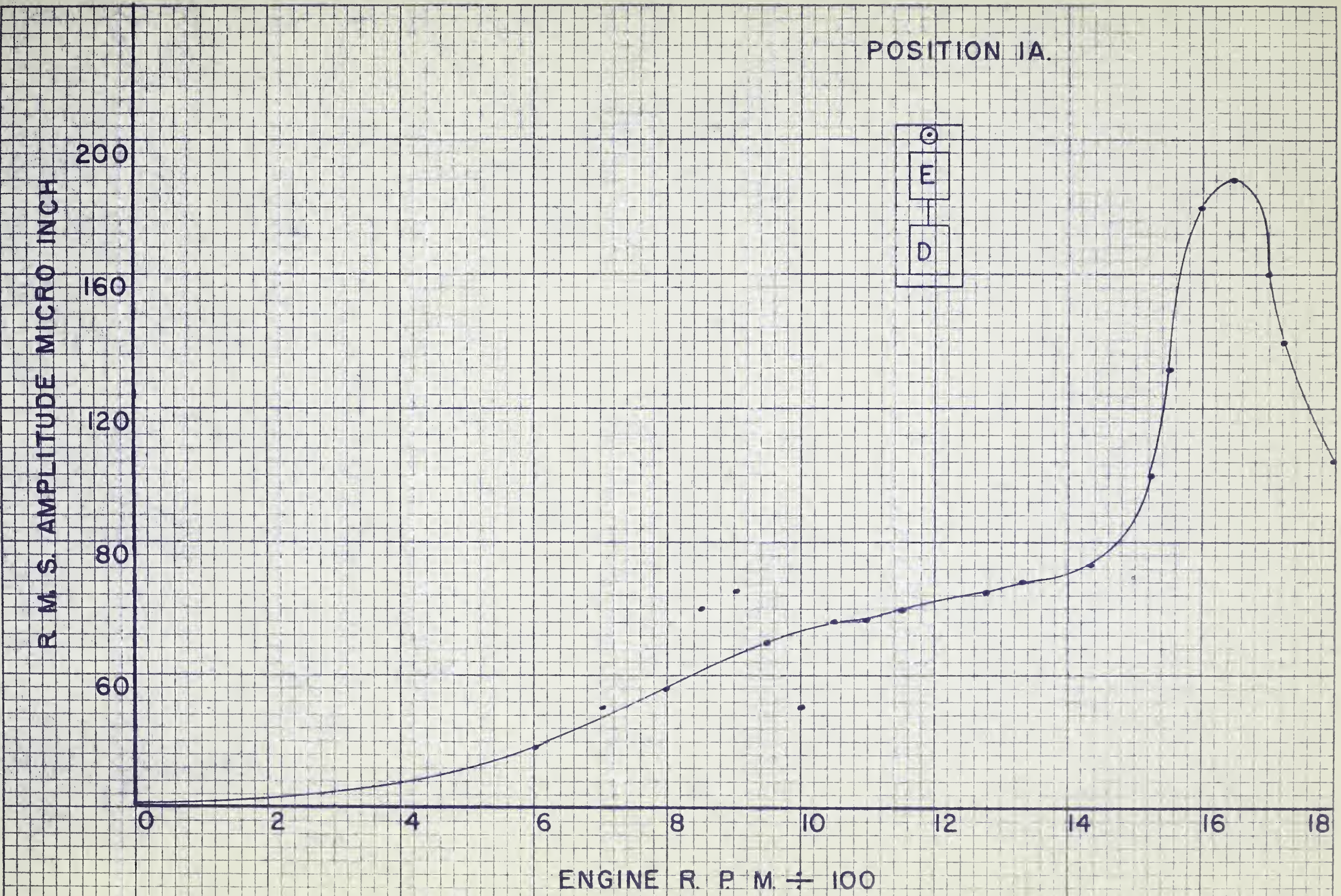

POSITION IA.
E
D
R. M. S. AMPLITUDE MICRO INCH
200
160
120
80
60
0
2
4
6
8
10
12
14
16
18
ENGINE R. P. M. ÷ 100

FIG. 10

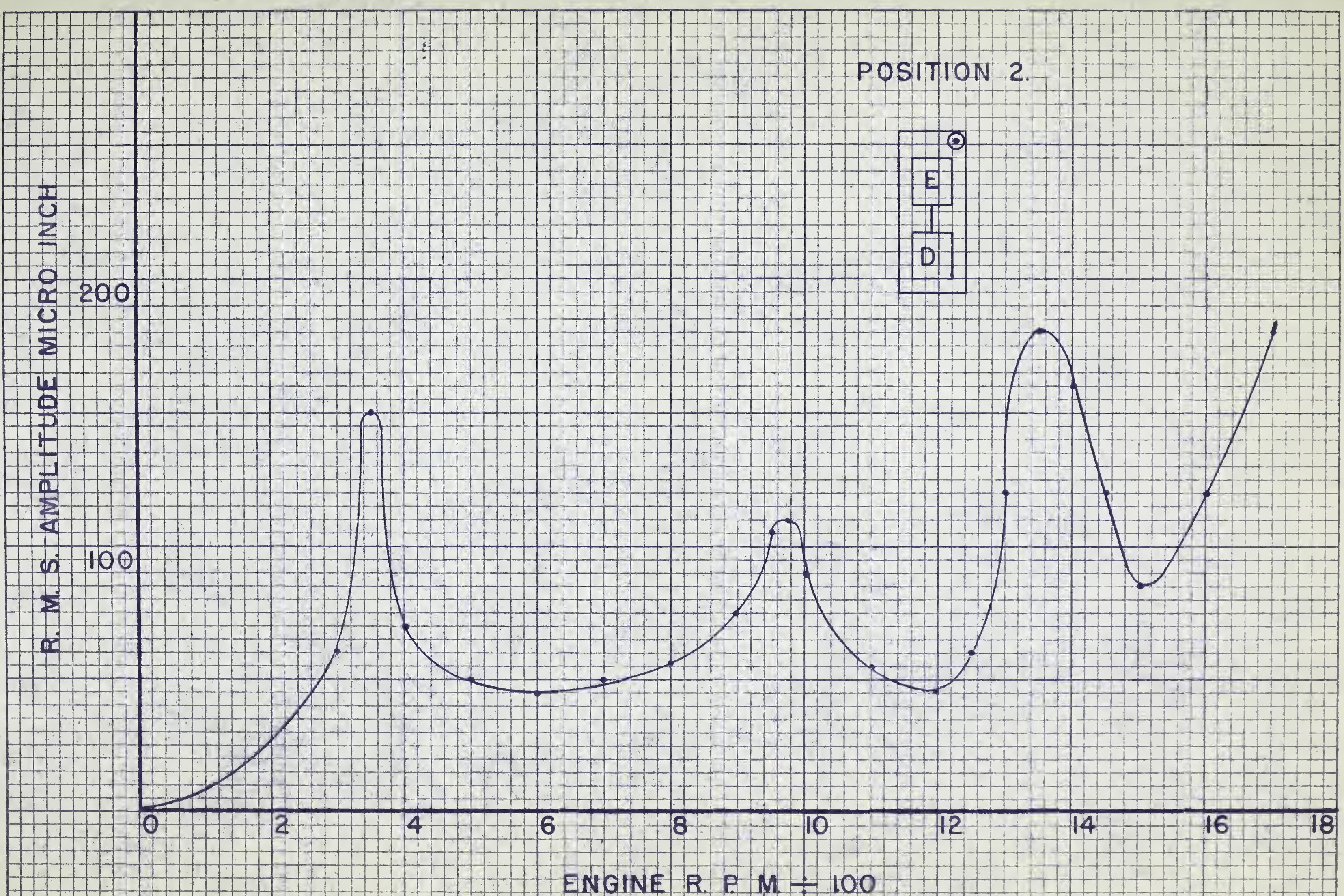
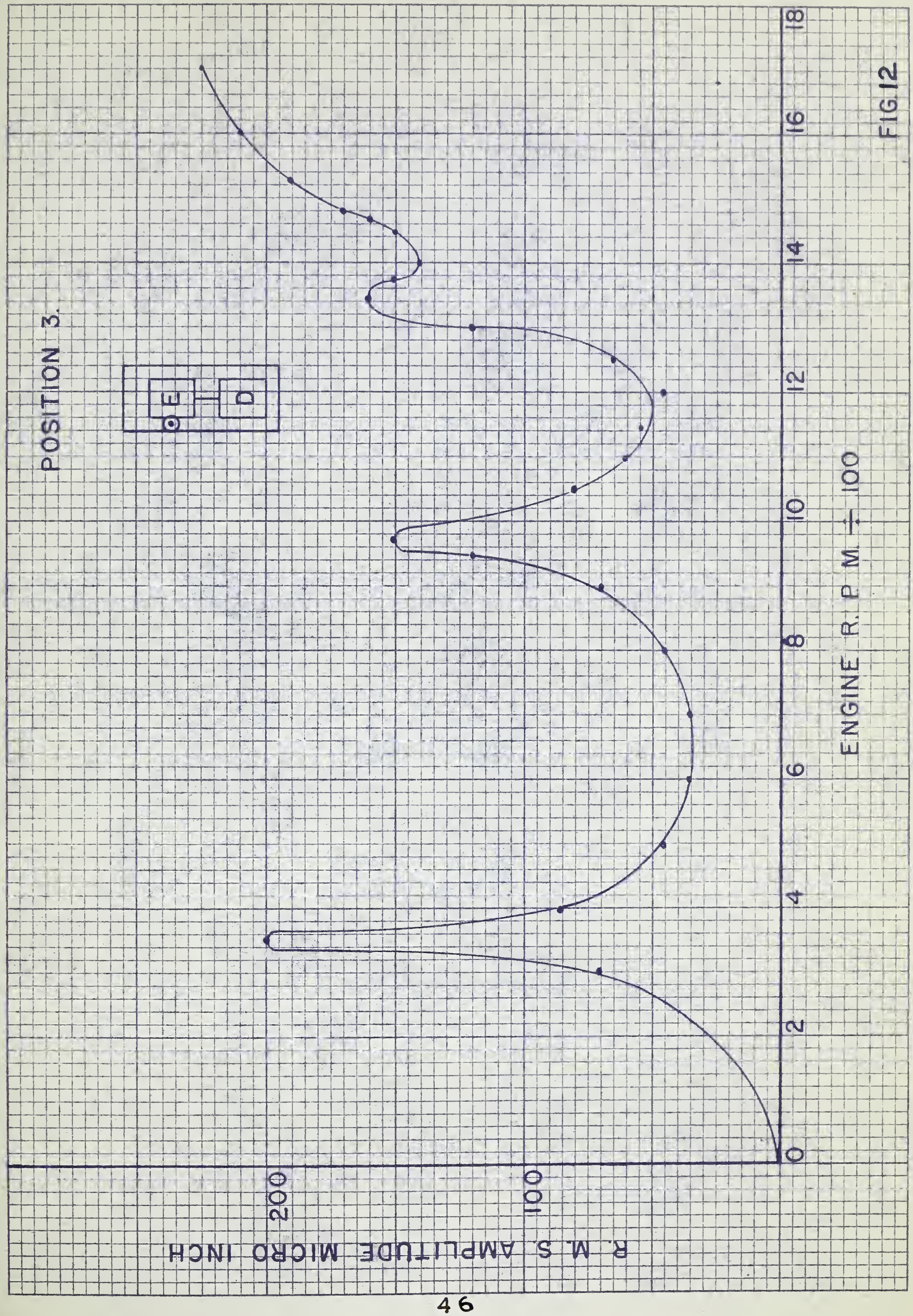

POSITION 2.
E
D
R. M. S. AMPLITUDE MICRO INCH
200
100
0
2
4
6
8
10
12
14
16
18
ENGINE R. P. M ÷ 100

FIG.11


POSITION 3.
E
D
R. M. S. AMPLITUDE MICRO INCH
100
200
0
2
4
6
8
10
12
14
16
18
ENGINE R. P. M. ÷ 100
FIG.12

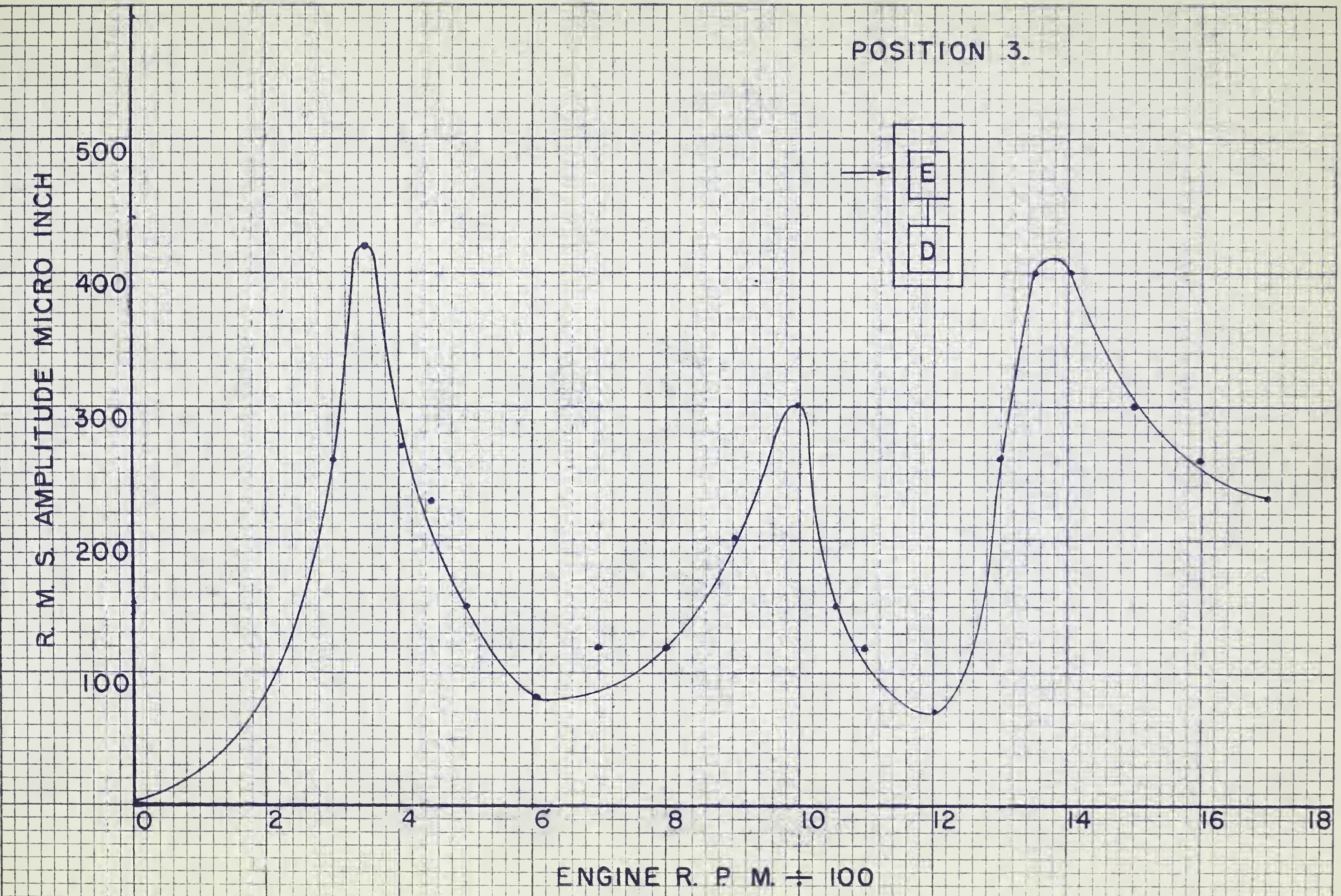

POSITION 3.
E
D
R. M. S. AMPLITUDE MICRO INCH
500
400
300
200
100
0
2
4
6
8
10
12
14
16
18
ENGINE R. P. M. ÷ 100

FIG. 13

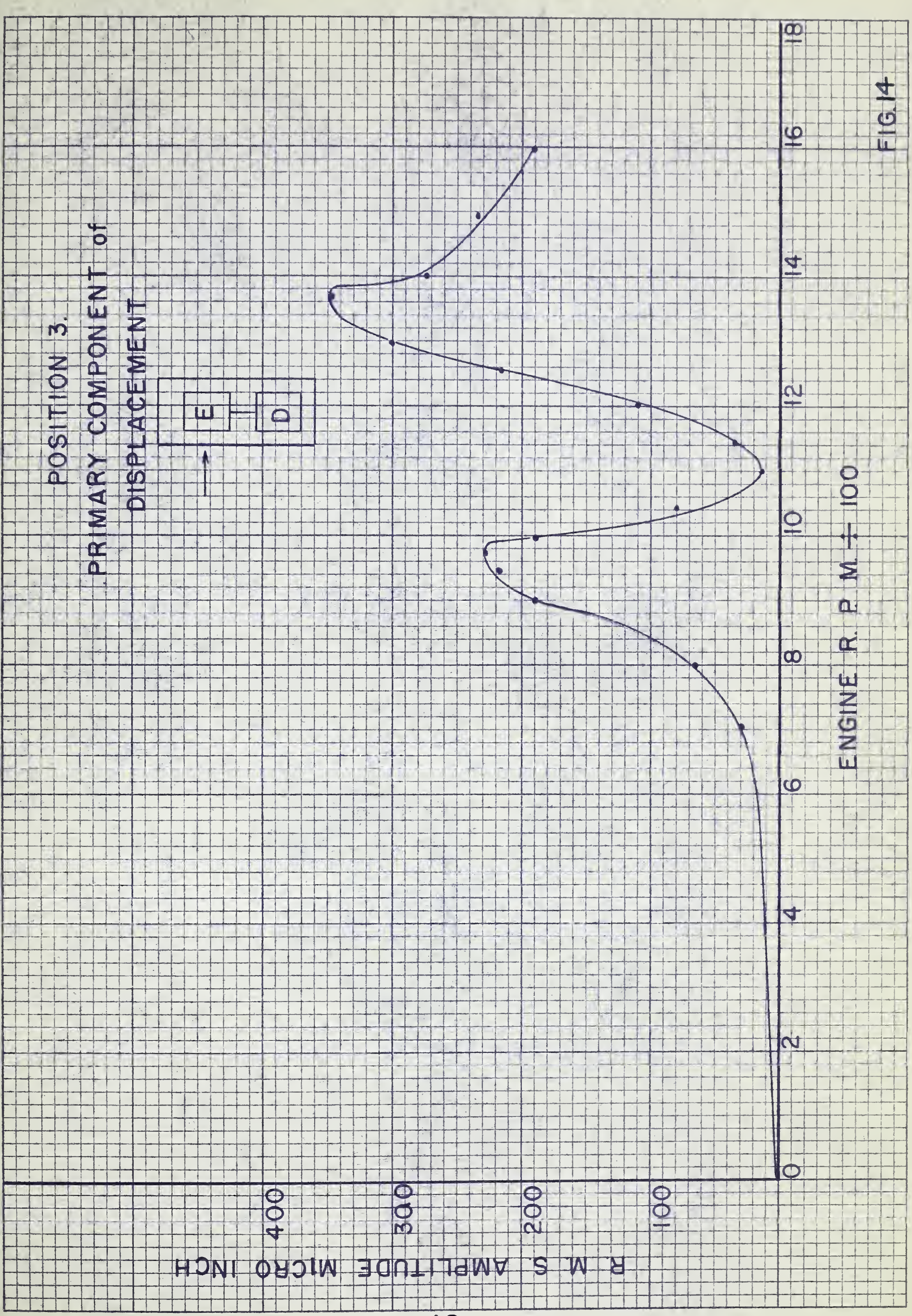

POSITION 3.
PRIMARY COMPONENT of
DISPLACEMENT
E
D
R. M. S. AMPLITUDE MICRO INCH
400
300
200
100
0
2
4
6
8
10
12
14
16
18
ENGINE R. P. M. ÷ 100

FIG. 14

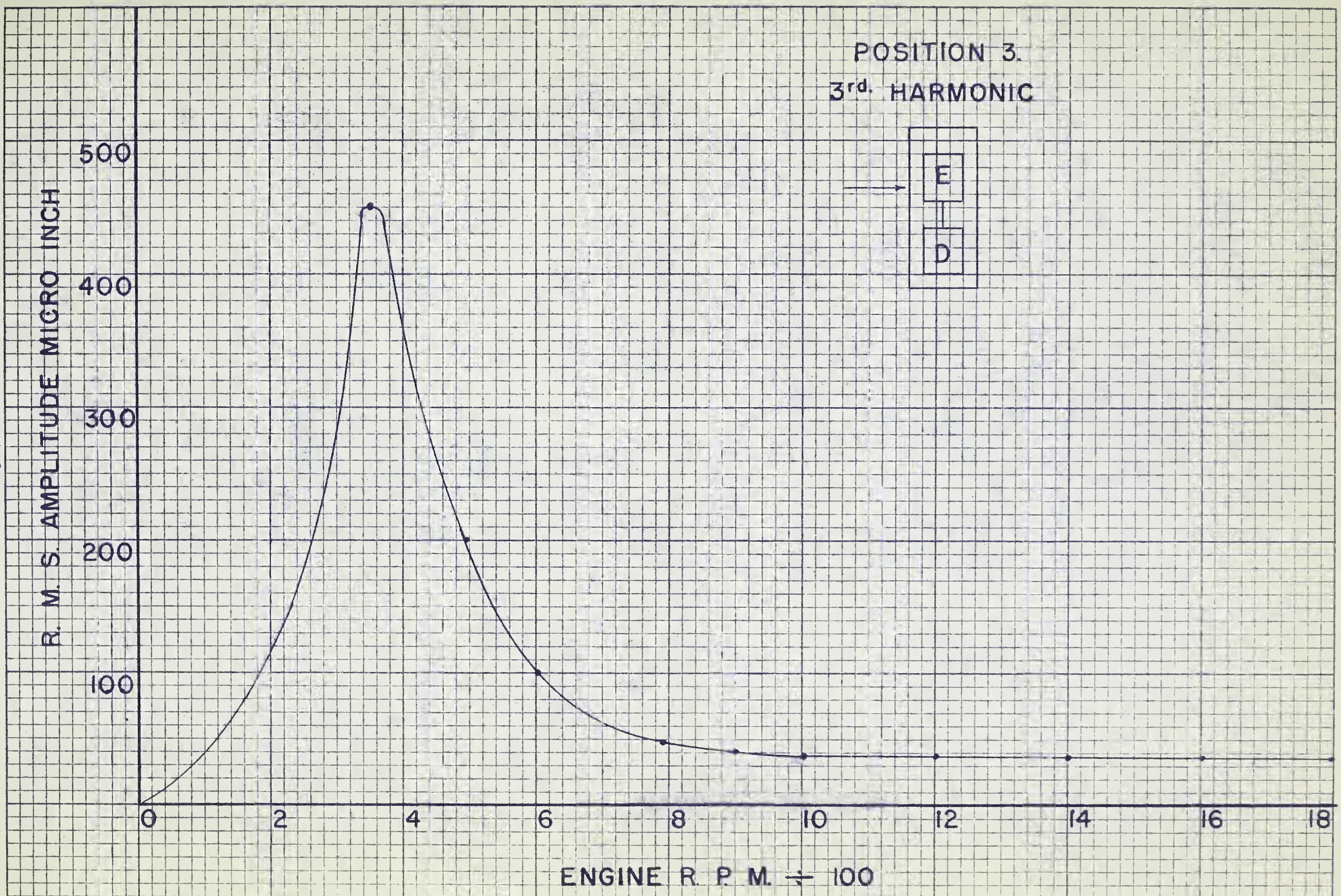

POSITION 3.
3rd. HARMONIC
E
D
R. M. S. AMPLITUDE MICRO INCH
500
400
300
200
100
0
2
4
6
8
10
12
14
16
18
ENGINE R. P. M. ÷ 100

FIG. 15

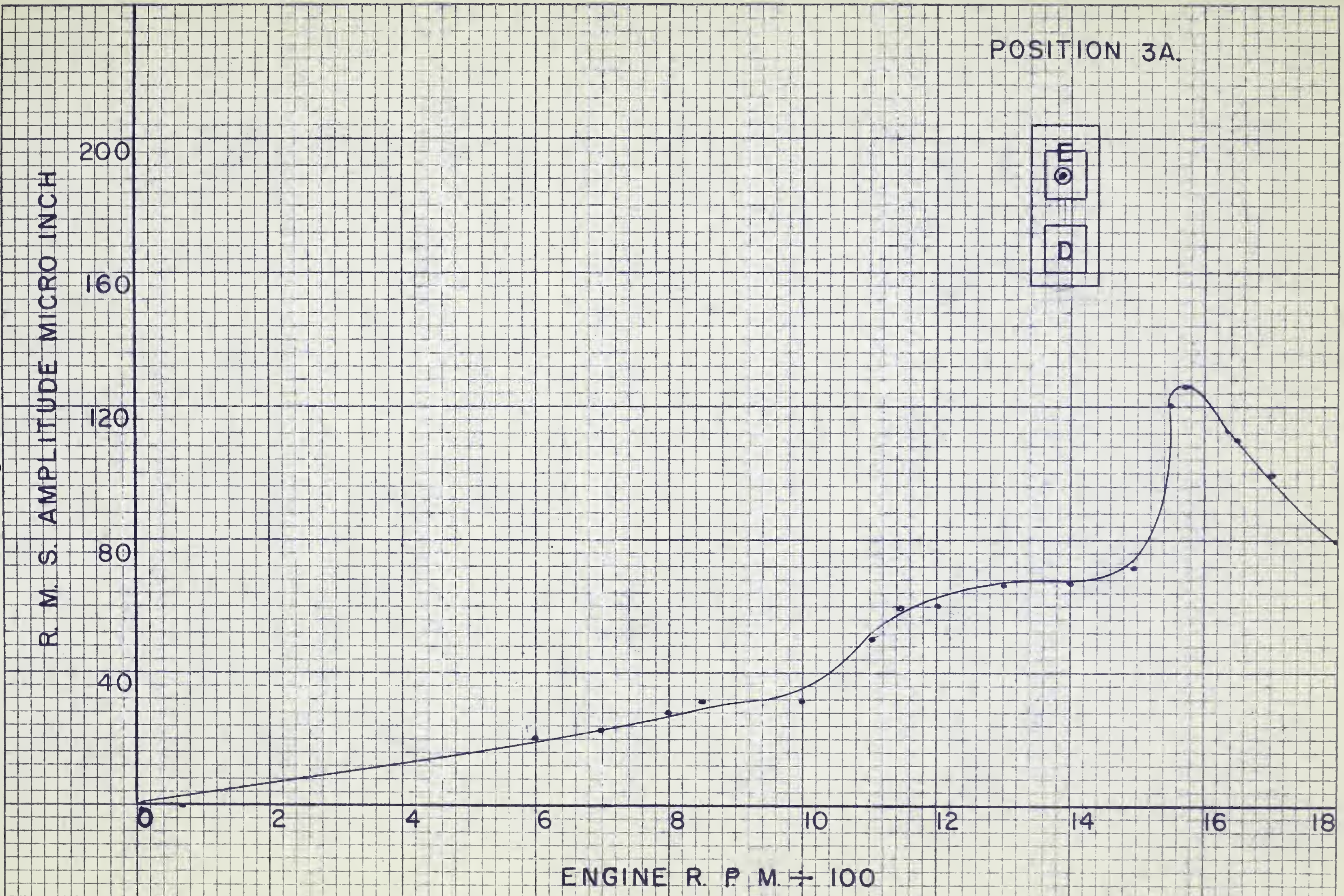

POSITION 3A.
E
D
R. M. S. AMPLITUDE MICRO INCH
200
160
120
80
40
0
2
4
6
8
10
12
14
16
18
ENGINE R. P. M. ÷ 100

FIG. 16

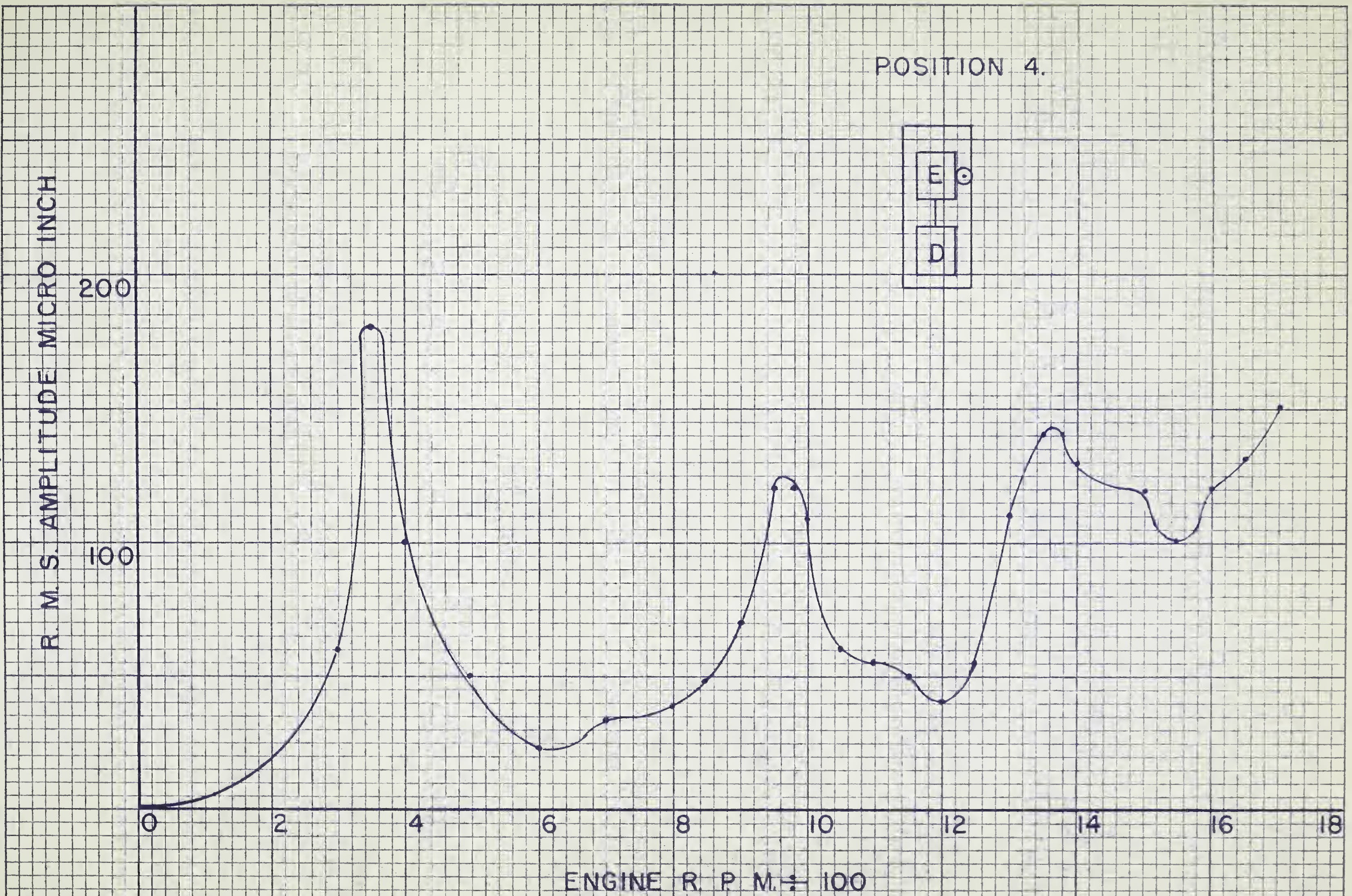

POSITION 4.
E
D
R. M. S. AMPLITUDE MICRO INCH
200
100
0
2
4
6
8
10
12
14
16
18
ENGINE R. P. M. ÷ 100

FIG. 17

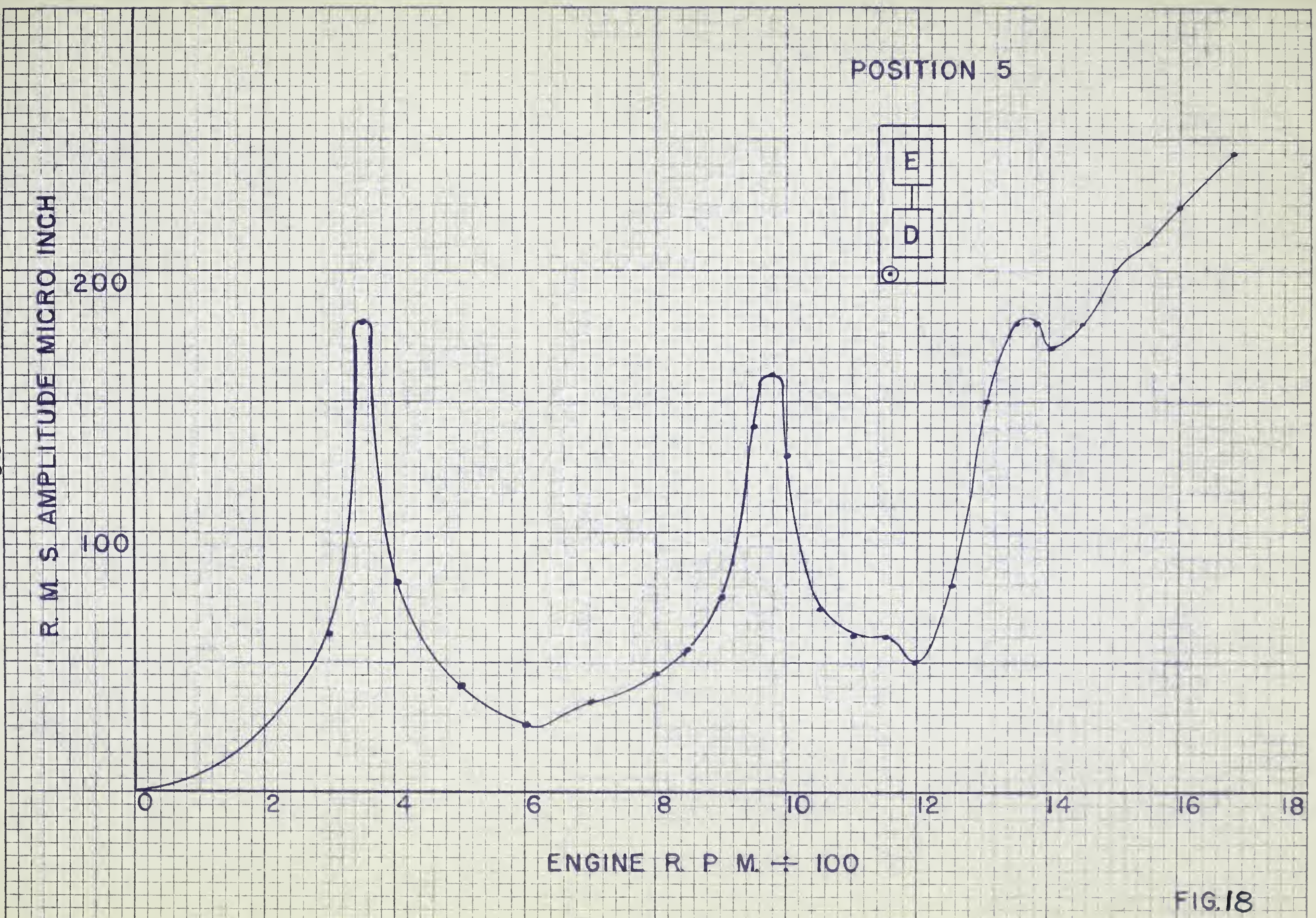
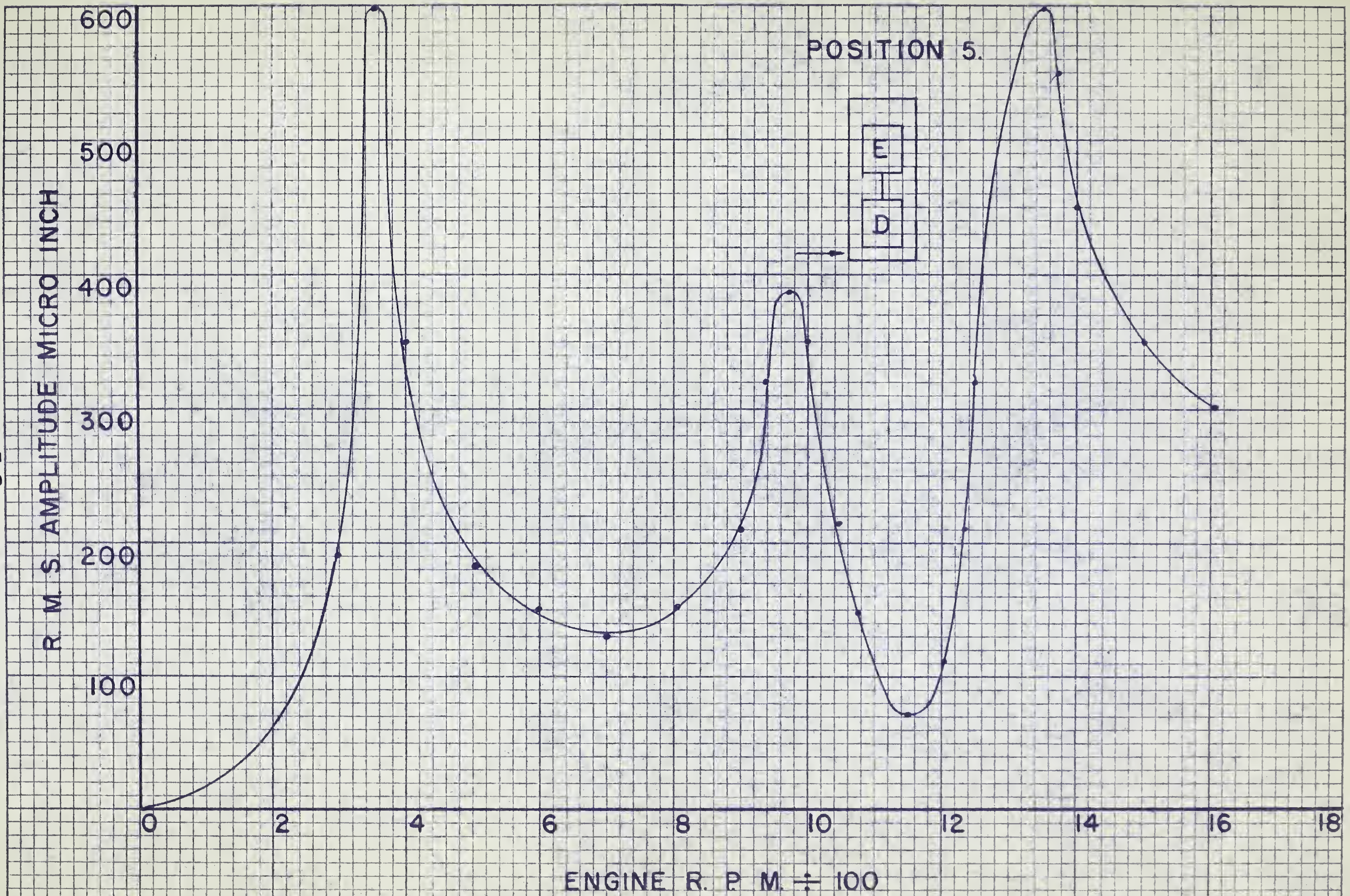
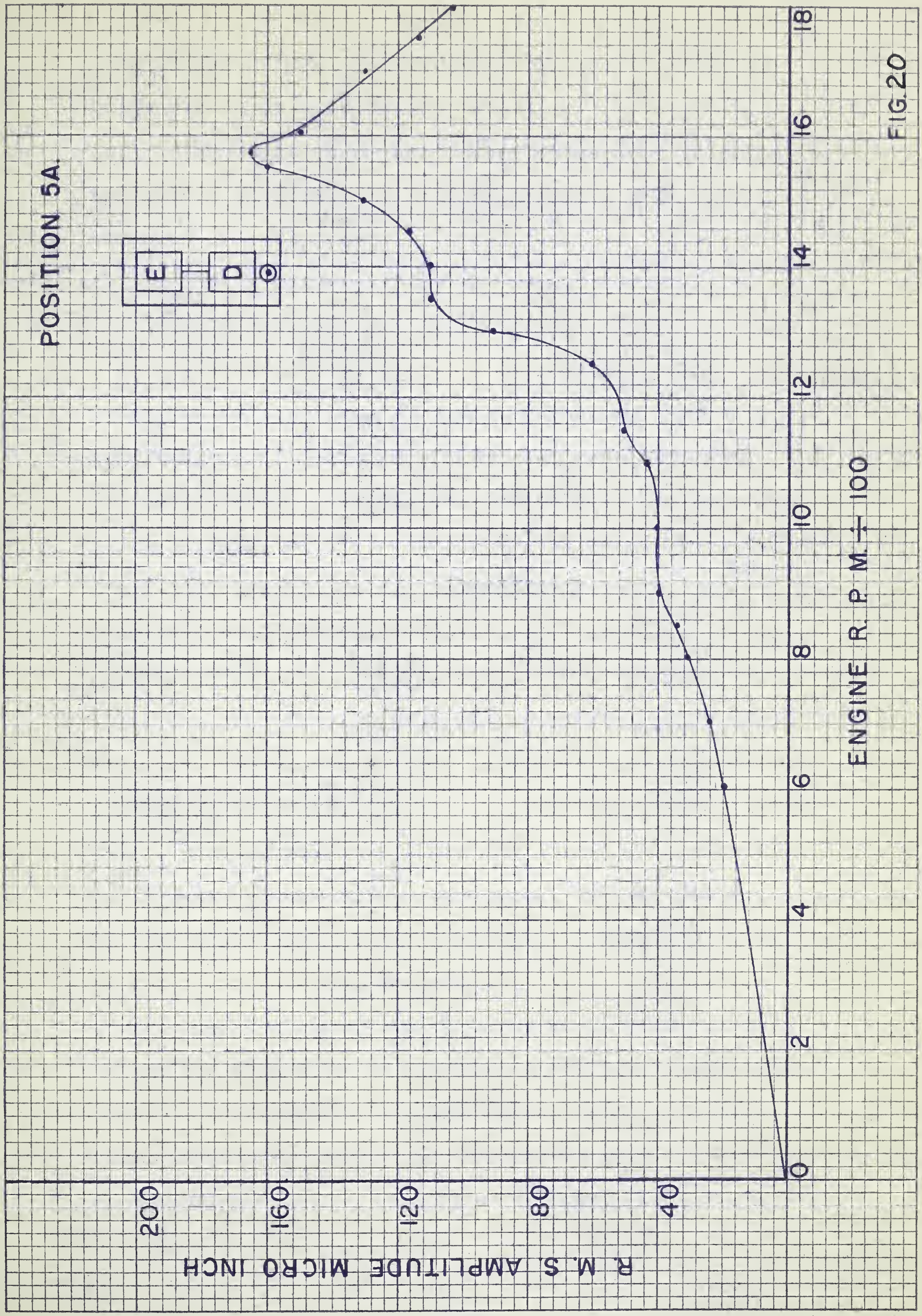

POSITION 5
E
D
R. M. S. AMPLITUDE MICRO INCH
200
100
0
2
4
6
8
10
12
14
16
18
ENGINE R. P. M. ÷ 100

FIG. 18



POSITION 5.
E
D
600
500
400
300
200
100
0
2
4
6
8
10
12
14
16
18
R. M. S. AMPLITUDE MICRO INCH
ENGINE R. P. M. ÷ 100

FIG. 19


POSITION 5A
E
D
R. M. S. AMPLITUDE MICRO INCH
200
160
120
80
40
0
2
4
6
8
10
12
14
16
18
ENGINE R. P. M. ÷ 100
FIG. 20

tion of the arrow. Similarly, a ringed dot means the graph refers to vertical vibration.

Examination of the curves shows that resonance is present at three engine speeds, below 1600 R.P.M., namely:-

350 R.P.M.

975 R.P.M.

1375 R.P.M.

Resonance is also present somewhere between 1600 R.P.M. and 1800 R.P.M., however, it was not possible to run the engine at speeds greater than 1600 R.P.M. for more than a few seconds and the accuracy of the graphs is uncertain above 1600 R.P.M. On the longitudinal centre line of the foundation the peaks of displacement amplitude at 350, 975 and 1375 R.P.M. are not evident for vertical vibration (Fig. 10, Fig. 16, Fig. 20). This suggests these resonance conditions are due to transverse vibration and rolling about an axis parallel to the longitudinal centre line. Displacement amplitudes for horizontal vibration are greater at the dynamometer end. This is evident if Fig. 9 and Fig. 19 are compared. A rotation about a vertical axis is thus present.

Application of the wave analyser at position 3 showed the wave to have only two significant components, namely, a fundamental harmonic and a third harmonic, (Fig. 14 and Fig. 15). The cathode ray oscilloscope provided verification of this at engine speeds above 600

R.P.M. as the wave form appeared to be symmetrical about a horizontal axis, indicating the absence of even harmonics. It is significant that the resonant peak at 350 R.P.M. is almost entirely due to the third harmonic. An apparently obvious assumption is that the third harmonic is due to the three power strokes per revolution of the engine. This cannot be correct, however, since the same amplitude - R.P.M. characteristics were obtained when the engine was motored. The third harmonic of the displacement must therefore be due to the rolling couple whose principal component has an angular frequency three times the engine angular frequency.

It appears that the natural frequency for vertical vibrations is between 1600 and 1800 c.p.m. This is indicated by Figs. 8, 10, 11, 12, 16, 17, 18 and 20.

The experimental results show that the amplitude - R.P.M. characteristics for the foundation are not similar to those for a damped spring supported mass. This is due to the various degrees of freedom and the periodic disturbing forces and couples which are not simple harmonic.

It is of interest to consider the dynamic soil pressure caused by the vertical vibration. The following calculations are highly approximate because of the lack of exact values for the dynamic properties of the soil. If the soil-vibrator dynamic value of G is taken as 1500 p.s.i. (which is considerably higher than

the static value but less than the value found from wave velocities) then from Eq. 2.4; the coefficient of dynamic subgrade reaction is:-

$$k' = \frac{2(1 + \nu)G}{m(1 - \nu^2)\sqrt{A}}$$

$$= \frac{2 \times (1 + .5) \times 1500 \times 144}{0.92 \times (1 - .25) \times 8.3 \times 12} \text{ lbs. per sq.ft.in.}$$

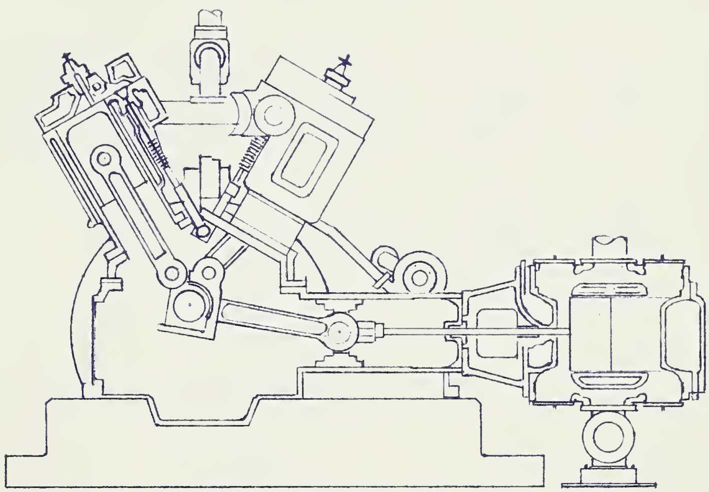
$$= 9{,}450 \text{ lbs. per sq.ft.in.}$$

This value should be regarded as an order of magnitude. The greatest vertical root-mean-square amplitude found was 0.00025 in. This means the greatest dynamic pressure was $9{,}450 \times 0.00025 \times \sqrt{2} = 3.35$ lbs. per sq. ft., which is negligible when compared to the static bearing pressure.

To conclude, the foundation must be considered satisfactory as the largest root-mean-square amplitude recorded is 600 micro inches, that is a double amplitude of $0.0006 \times 2\sqrt{2} = 0.0017$ inches. This is partly due to the low bearing pressure of 390 lbs. per sq. ft.

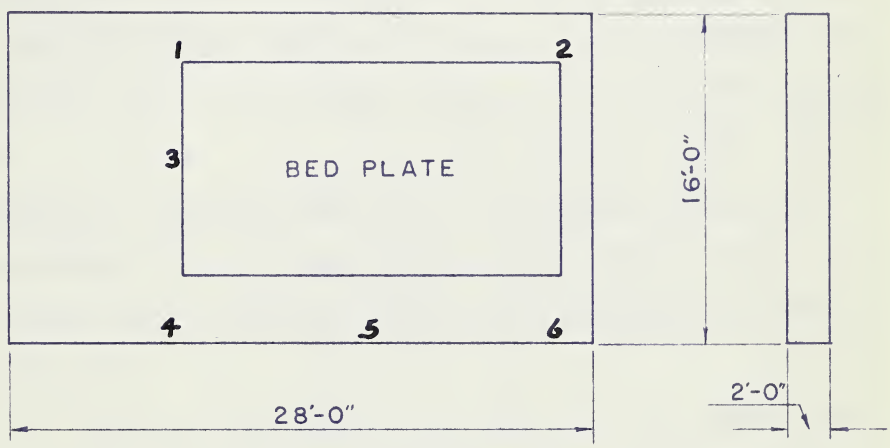
(b) A Vibration Study of a Compressor Foundation Before and After Modification.

The foundation of Imperial Oil Battery No. 40 at Leduc, Alberta was subject to excessive vibration, therefore it was decided to modify the foundation. Battery No. 40 consists of an Ingersoll Rand J.V.G. Gas Engine Compressor installation.



GAS ENGINE COMPRESSOR

FIG. 21



COMPRESSOR FOUNDATION

FIG. 22

Readings of displacement amplitudes were taken with the General Radio Company Vibration Meter before and after the modification.

Before the modification the foundation was a reinforced concrete slab as shown in Fig. 22. Other details were as follows:-

Weight of machine and foundation	537,600 lb.
Weight of concrete foundation	134,500 lb.
Weight of machine and bed plate	403,100 lb.
Load per sq. ft. of contact area	1200 lb. per sq. ft.
Soil	Clay

Machine -- Ingersol Rand J.V.G. gas engine compressor consisting of a V-6 gas engine $8\frac{1}{4}$ in. bore x 9 in. stroke and 3 cylinder horizontal compressor as shown in Fig. 21.

It was not possible to determine the masses of the moving parts of the machine in order to calculate the disturbing forces and moments. Qualitatively the disturbing effects are as follows:-

(1) Pitching couple about a transverse axis and with primary and secondary components. This couple is due to the V engine.

(2) Yawing couple about a vertical axis and with primary and secondary components. This couple is due to the three horizontal compressor cylinders.

The results of the displacement amplitude readings before and after the modifications are shown in Table 1. Before the modification was made, the

greatest component of vibration was in the transverse direction. Displacement amplitude of the transverse vibration varied linearly in the longitudinal direction. This means the vibration consisted mainly of translation of the centre of gravity and rotation about a vertical axis through the centre of gravity. Pitching and rolling about horizontal axes along with vertical displacement were also present, however, these components were small compared with the transverse vibration. The longitudinal component which was present despite the absence of disturbing forces in the longitudinal direction was due to coupling of the various degrees of freedom.

Results in Table 1 indicate that before the modification was made the natural frequency for both transverse and vertical vibrations was approximately 475 c.p.m., and the results of Table 2 show the natural frequency for yawing about a vertical axis was greater than 500 c.p.m. The term natural frequency used here refers to the lowest R.P.M. at which a resonance peak of an amplitude - R.P.M. curve occurs. It was observed during the tests that there was no resonance peak below 400 R.P.M.

Since the operating speed of the machine is between 400 and 500 R.P.M., it is evident that before the modification was made the foundation was always close to resonance when the machine was running.

The same method and the same values for the soil properties as in the previous study may be used to

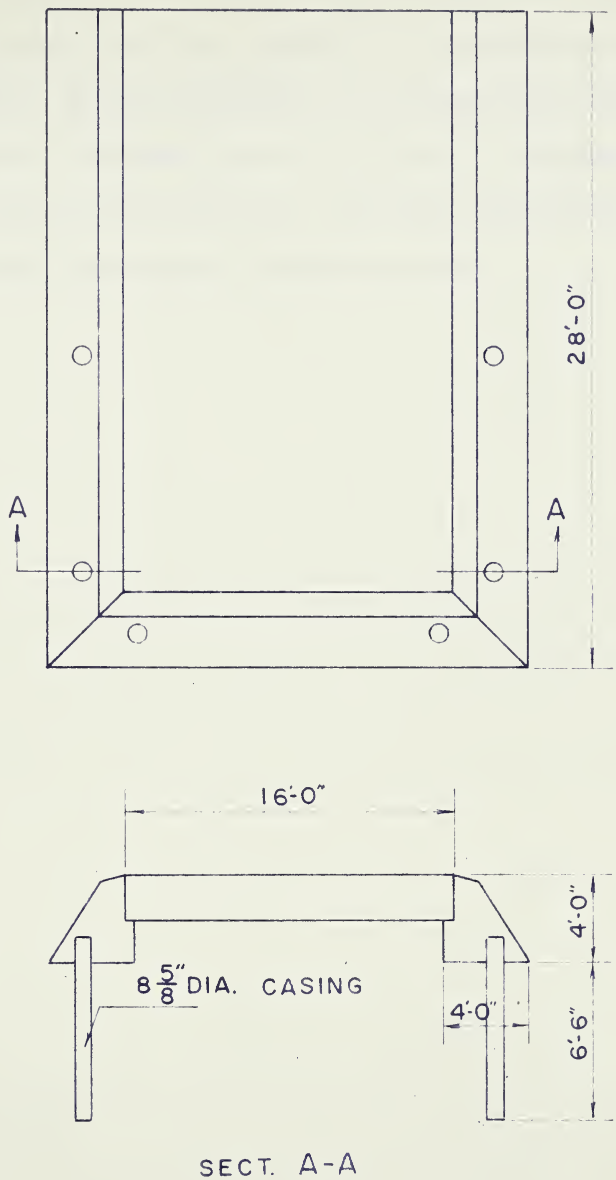
obtain an order of magnitude for the value of the dynamic soil pressure. A very approximate value for the dynamic modulus of subgrade reaction is found to be 4,000 lb. per sq. ft. in. and the greatest vertical amplitude recorded is 0.0014 in. giving a dynamic soil pressure of 5.0 lb. per sq. ft. This dynamic pressure is negligible when compared to the static bearing pressure.

The modification consisted of walls and piles which were added along the two long sides and the short side at the flywheel end as shown in Fig. 23. The weight of the concrete and piles which was added is 60,000 lb. making the total weight of the machine and foundation 597,600 lb. An increase in the transverse stiffness was desired hence the walls along the long sides and the piles.

A comparison of vibration readings taken before and after the modification was made indicate that amplitudes were reduced by a factor of one third to one tenth. Readings taken after the modication was made indicate that the natural frequencies for the transverse and vertical degrees of freedom are higher than 500 R.P.M.

It is evident that as well as an increase in the transverse stiffness the modification resulted in an increase in the vibrating mass, including the soil vibrating in phase with the foundation. An increase in the damping is also probable.

In order to explain the results of the modification an analogous single degree of freedom damped



MODIFIED FOUNDATION

FIG. 23

spring supported mass may be considered and Eq. 2.5 applied. If the vibrating mass, the stiffness and the damping constant are increased in the same proportion, Eq. 2.5 shows that the amplitude of displacement is reduced for a given frequency of disturbing force. Since the natural frequency was raised it is probable that the modification increased the stiffness more than it increased the equivalent vibrating mass.

TABLE 1

Transverse Vibration (y direction)

	RPM	Double Amplitude Before	After
Positions 1, 3 & 4	400	0.0055 in	0.0012 in
	450	0.0073 in	0.0013 in
	500	0.0056 in	0.0015 in
Position 4	400	0.0055 in	0.001 in
	450		0.0013 in
	500	0.0078 in	0.002 in
Positions 2 & 6	400	0.007 in	0.0011 in
	450	0.012 in	0.0013 in
	500	0.011 in	0.0017 in

Vertical Vibration (z direction)

	RPM	Double Amplitude Before	After
Position 1	400	0.0015 in	0.0002 in
	450	0.0027 in	0.00027 in
	500	0.0021 in	0.0003 in
Position 2	400	0.0028 in	0.00015 in
	450		0.00024 in
	500		0.0003 in
Position 4	400	0.0014 in	0.0003 in
	450	0.0019 in	
	500	0.0016 in	
Position 6	400	0.001 in	0.00015 in
	450	0.0015 in	
	500	0.0012 in	0.00013 in

Longitudinal Vibration (x direction)

	RPM	Double Amplitude Before	After
Position 1	400	0.002 in	negligible
	450	0.0021 in	"
	500	0.0022 in	"

TABLE 1 (continued)

Longitudinal Vibration (x direction)

	RPM	Double Amplitude Before	After
Position 2	400	0.002 in	negligible
	450		"
	500	0.002 in	"
Position 4	400	0.002 in	negligible
	450	0.003 in	"
	500	0.002 in	"
Position 5	400		negligible
	450	0.0025 in	"
	500	0.0022 in	"

TABLE 2

Approximate Double Amplitude of Rotation

About a Vertical Axis (Yawing)

RPM	
400	0.0008°
450	0.0013°
500	0.0014°

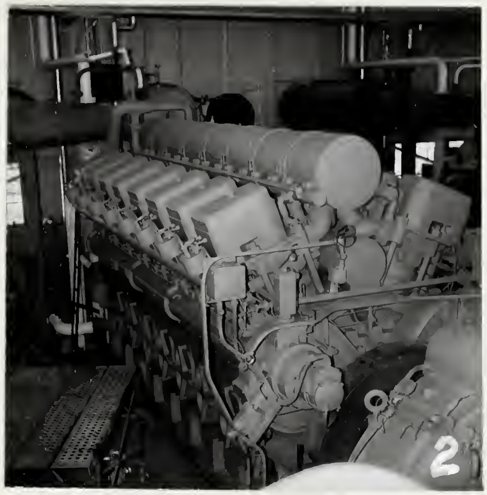
(c) A Study of Successful Gas Engine Foundations

This is a vibration study of the machine foundations at No. 5 Station of the West Coast Transmission Co. at Australian, British Columbia. The plant consists mainly of four De Laval centrifugal compressors, each driven by a V-16 Nordberg Supair-thermal Gas Engine (Fig. 24) through an 11:1 step up gear box. Each compressor along with its gear box and engine is on a separate foundation as shown in Fig. 26.

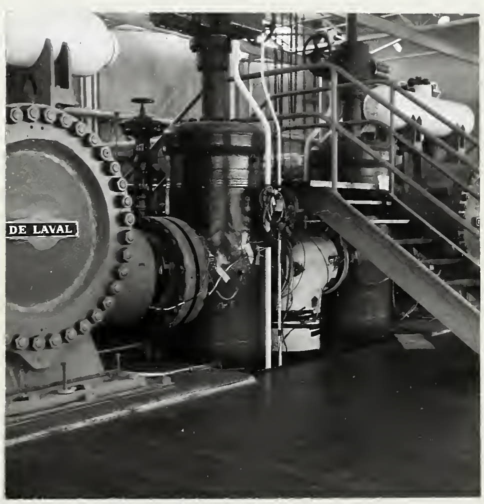
The bearing pressure on the clay soil is 1,600 lb. per sq. ft. According to Newcomb (2) this is a very reasonable bearing pressure. Any vibration of the foundations is caused mainly by the engines as the compressors operate at 5,500 R.P.M. which is probably several times the natural frequency of the foundations. Disturbing forces from the compressors are negligible since the rotors are statically and dynamically balanced.

Measurements taken with the General Radio Company Type 761 vibration meter showed that the vibration level was very low and these foundations may be considered extremely successful from the vibration standpoint. This is probably due partly to the fact that V-16 engines are inherently balanced having no disturbing forces or moments with first, second or third harmonics. It was possible to stand a nickel on end at any point on the bed plate of an engine when it was running at from 425 to 500 R.P.M.

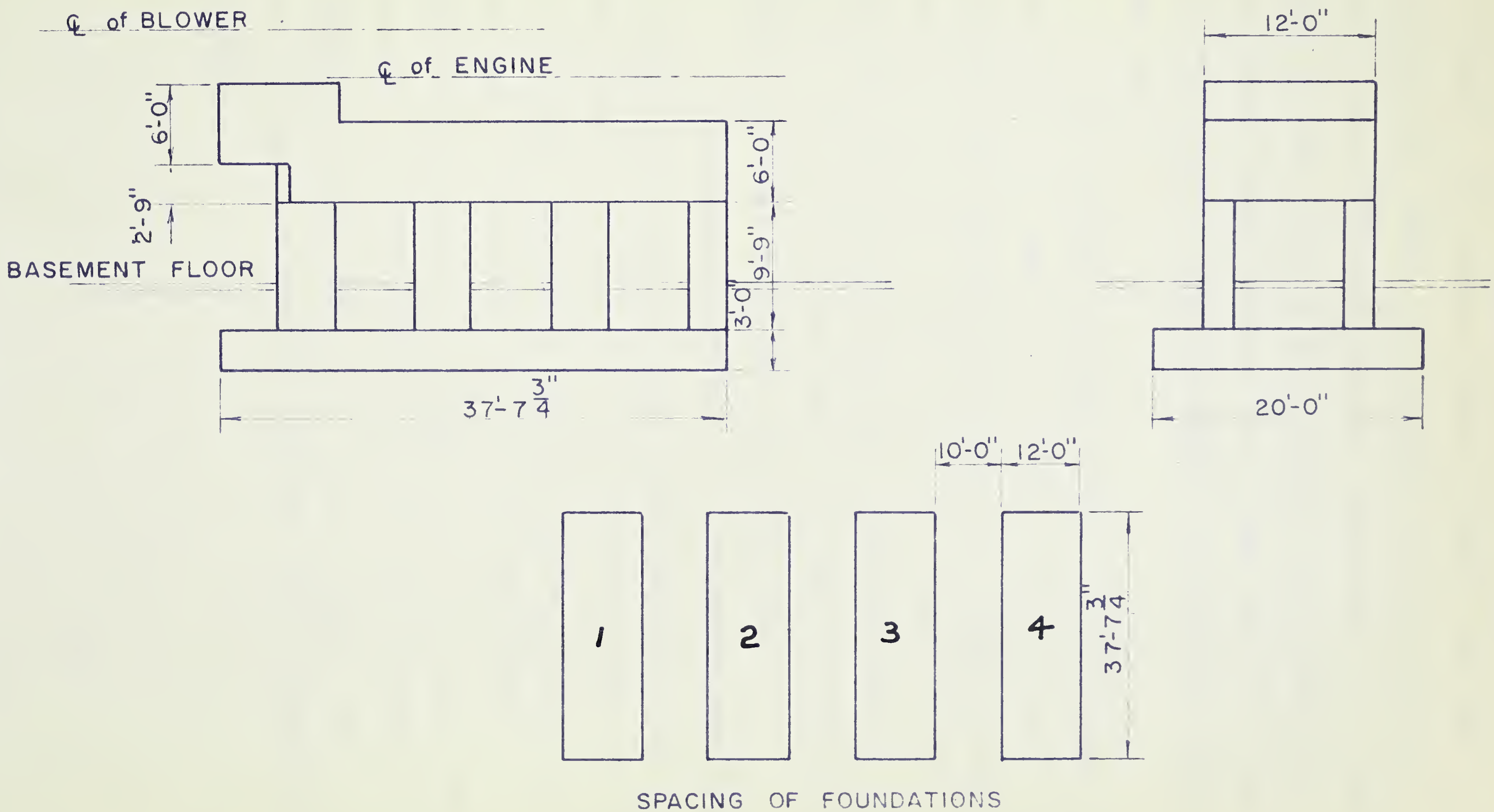
Table 3 gives the double amplitudes of the dis-



V-16 GAS ENGINE Fig. 24



COMPRESSOR AND PIPING Fig. 25



V-16 GAS ENGINE FOUNDATION

FIG. 26

placements. One engine only was running when the readings were taken. The readings indicate that an engine-compressor base vibrates essentially as a rigid body. At the level of the basement floor transverse displacement amplitudes on the columns were less than the lower limit of the instrument (16 micro-in. root-mean-square).

The readings in Table 3 show that the bases vibrate with four degrees of freedom; translations in the vertical and transverse directions and rotations about a vertical axis and a transverse axis, except for No. 4 which has no rotation about the transverse axis at 450 R.P.M. Displacement amplitudes are very low the greatest double amplitude being less than 0.001 in.

The centre of gravity of a foundation and machine is fairly high above the base which suggests a low natural frequency for rotation about a transverse axis. It may be fortunate, therefore, that the engines produce no transverse disturbing forces or rolling moments. A factor which probably influences the vibrational characteristics of the foundations is the restraint due to the compressor piping (Fig. 25). This restraint tends to resist rolling or transverse movement and is therefore desirable.

TABLE 3

Foundation No. 1
(Compressor Coupled)

RPM	Direction	Position	Amplitude	Remarks
450	Transverse	Compressor end	0.00014 in.	Linear variation in longitudinal direction.
450	"	Engine end	0.00084 in.	
425	"	Compressor end	0.00011 in.	Linear variation in the longitudinal direction.
425	"	Engine end	0.00067 in.	
450	Vertical	Compressor end	0.00005 in.	No variation in transverse direction.
450	"	Engine end	0.00007 in.	Linear longitudinal variation.

Vibration in longitudinal direction was negligible.

Foundation No. 2
(Compressor Coupled)

RPM	Direction	Position	Amplitude	Remarks
500	Transverse	Compressor end	0.00025 in.	Linear variation in longitudinal direction.
500	"	Engine end	0.00085 in.	
500	Vertical	Compressor end	0.00012 in.	No variation in transverse direction.
500	"	Engine end	0.00025 in.	Linear longitudinal variation.

Vibration in longitudinal direction was negligible.

TABLE 3 (continued)

Foundation No. 3
(Compressor Coupled unless otherwise stated)

RPM	Direction	Position	Amplitude	Remarks
400	Transverse	Compressor end	0.000035 in.	Compressor uncoupled and no variation in the transverse direction.
400	Vertical	Compressor end	0.000030 in.	
400	Vertical	Engine end	0.000050 in.	
425	Transverse	Compressor end	0.00028 in.	Linear variation in longitudinal direction.
425	"	Engine end	0.00022 in.	
425	Vertical	Compressor end	0.00022 in.	No variation in transverse direction. Linear variation in longitudinal direction.
425	"	Engine end	0.00028 in.	

Vibration in longitudinal direction was negligible.

Foundation No. 4
(Compressor Coupled)

RPM	Direction	Position	Amplitude	Remarks
425	Transverse	Compressor end	0.00020 in.	Linear variation in longitudinal direction.
425	"	Engine end	0.00040 in.	
450	Transverse	Compressor end	0.00020 in.	Linear variation in longitudinal direction.
450	"	Engine end	0.00060 in.	
450	Vertical	All	0.00007 in.	

Vibration in longitudinal direction was negligible.

---

6. DESIGN CONSIDERATIONS

The determination of the natural frequencies of all the modes of vibration of the foundation of a low speed machine is seldom possible by analytical methods. This is mainly due to the unavailability of accurate values of the dynamic properties of the soil and values for the moments of inertia of the machine about certain axes. It follows that the designer must consider certain factors in a qualitative manner. This may lead to overdesign but at present appears to be unavoidable.

Successful foundations for low speed machines usually have natural frequencies which are considerably higher than the machine frequency (2). It is, therefore, desirable that designers strive to achieve foundation natural frequencies which are several times the operating frequency of the machine.

Two basic considerations of a designer of machine foundations are the characteristics of the machine and the characteristics of the soil upon which the foundation is to rest.

The important characteristics of a machine from a foundation designer's standpoint are the weight, the operating speed and the unbalance forces and couples. The weight and operating speed are usually known by the foundation designer. However, Tschebotarioff (11) has commented on the reluctance of machine manufacturers to supply data on the unbalance effects. This lack of

cooperation by the manufacturers is not serious as any competent engineer can determine at least the nature if not the magnitude of the unbalance effects by consulting references (14) and (17).

When unbalance effects are known qualitatively certain preliminary assumptions may sometimes be made regarding the geometry of the foundation. For example, if the machine has rolling couples or transverse disturbing forces it is desirable that the ratio of the width of the bearing area to the height of the centre of gravity be as high as possible. Rolling couples and transverse disturbing forces tend to excite rolling vibration about an axis parallel to the crankshaft axis. Therefore for low speed machine foundations the natural frequency for the rolling degree of freedom should be as high as possible if rolling couples and transverse disturbing forces are present.

An expression for the natural frequency of rolling vibrations about an axis in the bearing area and parallel to the crankshaft axis is as follows:

$$f = \frac{1}{2\pi}\sqrt{\frac{K}{I_v}}$$

where I_v = mass moment of inertia of machine and foundation about axis of rotation.

K = restoring moment per radian of rotation.

This equation is analogous to Eq. 2.1 and can-

not be expected to give accurate results as the effect of the vibrating soil is neglected. However, it may be used to show the importance of the ratio of the width of the bearing area to the height of the centre of gravity.

It can be shown that for a rectangular bearing area the restoring moment per radian of rotation is given by:-

$$K = k' \frac{ba^3}{12}$$

$$= k'I'$$

where k' = dynamic modulus of subgrade reaction.

a = width of bearing area

b = length of side parallel to axis of rotation.

I' = second moment of area of bearing area about the axis of rotation.

The natural frequency is then given by:

$$f = \frac{1}{2\pi} \sqrt{\frac{k'ba^3g}{12r^2W_v}}$$

where W_v = weight of machine and foundation.

r = radius of gyration of machine and foundation about the axis of rotation.

This indicates that in order to keep the natural frequency high the bearing area should be as wide as possible and the height of the centre of gravity as low as possible.

If no rolling couples or transverse forces are present, a higher centre of gravity may be allowable. It must be remembered, however, that certain modes of vibration may be capable of existence without unbalance forces or couples to excite them. This is due to coupling of the various modes.

The higher harmonics of the disturbing forces and couples should be considered since the frequency of the dominant component of the vibrations of some machine foundations corresponds to twice the engine speed (20). Effects of resonance with the secondary component of a disturbing force or couple are unlikely to be as serious as those due to resonance with the primary component, since, for a given engine speed, the amplitude of the secondary component is seldom greater than two fifths that of the primary component (17).

Nevertheless, none of the natural frequencies of the foundation should be close to twice the engine frequency.

Unbalance forces produced by an engine or compressor are nearly always small compared with the weight of the machine and foundation. This indicates the non-linear vibrational characteristics are usually not of great concern to the foundation designer.

When the dynamic properties of the soil are known from the results of soil vibrator tests, the natural frequency of the vertical mode of vibration

may be found from one of the methods outlined in Part 2 of this thesis.

A knowledge of wave velocities in soils is useful if various soils are to be compared. For a given foundation geometry and bearing pressure the natural frequency is a function of the velocity of one of the wave types. Table 1 of reference 1 shows that the higher the velocity of Rayleigh waves the higher is the natural frequency. This suggests that although there is at present no direct method of determining the natural frequency from a wave velocity along with foundation weight and bearing pressure, wave velocities can be useful for qualitative considerations.

Terzaghi (4) mentions that the coefficient of dynamic subgrade reaction is considerably higher than the coefficient of subgrade reaction found from static tests. Consequently the results of static tests should be used with caution by machine foundation designers.

Morse (18) recommends a certain minimum weight per brake horsepower for the foundations of various machines. This is an unsatisfactory basis upon which to design as care must be taken to insure that the bearing pressure is considerably less than the allowable bearing pressure for static loads. A foundation with a large mass does not necessarily have a low amplitude of vibration unless the bearing pressure is low. The damped spring supported mass analogy indicates

that increasing weight, only, does not always decrease the amplitude of vibration. A low bearing pressure insures a high ratio of stiffness to vibrating mass and a low amplitude of vibration.

The height of the water table and the probable variation of this height should be considered by the foundation designer since the natural frequency of the foundation increases as the water table is raised (3).

For cases where it is not possible to keep within a desirable bearing pressure piles may be used. The use of piles for machine foundations is outside the scope of this thesis and reference (19) may be consulted.

The most convenient and practical methods for determining the vertical natural frequencies of foundations are given by Eq. 2.2 and Eq. 2.6. Eq. 2.6 may prove to be more useful since it involves only one unknown, namely β. A reliable value of β for sand is given by Eq. 2.7:-

$$= \frac{1.7}{m(1-\nu^2)\sqrt{A}}$$

Further investigation is required to obtain a reliable expression for β for clay, however, β for clay may tentatively be taken as

$$\beta = \frac{2.2}{m(1-\nu^2)\sqrt{A}}$$

A knowledge of the dynamic properties of the soil is necessary regardless of the method used to obtain the natural frequency.

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APPENDIX

Derivation of Eq. 2.7

Notation is the same as in Section (b) of Part 2. Consider the model shown in Fig. 3 to be loaded with a dynamic load P uniformly distributed over the surface area A. With the same assumptions as in Section (d) of Part 2 the dynamic stress at depth Z is then given by:

$$\sigma_z = \frac{P}{s}$$

$$= \frac{P}{abe^{\beta z}}$$

Strain at depth Z is then given by:-

$$\epsilon_z = \frac{P}{E'abe^{\beta z}}$$

Let W' = deflection at Z = 0

$$W' = \int_0^\infty \frac{P\,dz}{E'abe^{\beta z}}$$

$$= \frac{P}{E'\beta A}$$

A value for the deflection at Z = 0 is also given by Eq. 2.3

$$W = \frac{mP(1-\nu^2)}{E'\sqrt{A}}$$

Put W = BW'

Where B is a constant to be evaluated for different

soils.

$$\frac{BP}{E'\beta A} = \frac{mP(1-\nu^2)}{E'\sqrt{A}}$$

$$\therefore \beta = \frac{B}{m(1-\nu^2)\sqrt{A}}$$

The constant B may be found from the results of various vibrator tests. For example, a circular vibrator with area 10.8 sq. ft. and weight 6,050 lbs. resting on dry medium sand was found to have a natural frequency of 22 c.p.s. (1). Assuming G = 2,000 p.s.i., ρ = 100 lb. per cu. ft. and ν = 1/3, then from Eq. 2.6

$$22 = \frac{1}{2\pi}\sqrt{\frac{2x2{,}000x144x1.3x10.8x32.2x\beta}{\frac{100x10.8}{\beta} + 6050}}$$

$$\therefore \quad = 0.6$$

$$\therefore \quad 0.6 = \frac{B}{\sqrt{A}(1-\nu^2)}$$

$$\therefore \quad B = 1.75$$

The best average value of B for sand was found to be 1.7.

A similar procedure was used to determine a value of B = 2.2 for clay. However, this value cannot be considered as reliable as that for sand since insufficient test data was available to provide a check.